

Hofmann–Streicher lifting of fibred categories

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Background: Universes & Models

Universes (\mathcal{U})

- In set theory (Grothendieck): A tool for controlling sizes of categories.
- In type theory (Martin-Löf): A type of types.

By the mid 1990s, Martin-Löf type theory had acquired several credible notions of *categorical model*.

Motivating question

Are models of type theory stable under standard categorical constructions like forming presheaf categories?

The H-S construction answers this for presheaves. Given a universe $p: \tilde{\mathcal{U}} \to \mathcal{U}$ in **Set** and a \mathcal{U} -small category C, Hofmann and Streicher constructed a universe $p_{\hat{C}}: \tilde{\mathcal{U}}_{\hat{C}} \to \mathcal{U}_{\hat{C}}$ in the presheaf category \hat{C} .

The lifted universe is defined as follows, according to a reformulation by Awodey (2024):

$$\begin{split} \mathcal{U}_{\hat{C}}\left(c\right) &= \mathsf{Cat}\left(\left(C/c\right)^{op},\mathcal{U}\right)\\ \tilde{\mathcal{U}}_{\hat{C}}\left(c\right) &= \mathsf{Cat}\left(\left(C/c\right)^{op},\tilde{\mathcal{U}}\right) \end{split}$$



H-S Lifting and Universal Properties

While universes aren't defined by a universal property, the H-S lifting construction *preserves* universal properties when they exist.

Example: Subobject Classifiers

The subobject classifier in **Set** is the map $\{1\} \hookrightarrow 2$. Applying the Hofmann–Streicher lifting to **2** yields the subobject classifier $\Omega_{\hat{C}}$ in the presheaf topos \hat{C} :

 $\Omega_{\hat{C}}(c) = \operatorname{Cat} \left((C/c)^{op}, \mathbf{2} \right)$ $= \operatorname{Sieve}_{C}(c)$



A Functorial View (Awodey, 2024)

Awodey showed that the H-S lifting is the action of a functor, the **categorical nerve** ν_C .

Insight 1: An Adjunction

The nerve ν_C : $\mathbf{cat}_s \to \hat{C}$ is the right adjoint to the category of elements functor $\int_C : \hat{C} \to \mathbf{cat}_s$.

$$\hat{C} \xrightarrow[]{\mathcal{L}}{\overset{\int_{\mathcal{C}}}{\underset{\nu_{\mathcal{C}}}{\perp}}} \mathsf{cat}_{s}$$

It is defined as $\nu_{C}(D)(c) = \operatorname{cat}_{s}(C/c, D)$.



Insight 2: The Action on a Universe

The Hofmann–Streicher lifting of a universe $\tilde{\mathcal{U}} \to \mathcal{U}$ in **Set** to \hat{C} is obtained by applying the categorical nerve to the arrow $\tilde{\mathcal{U}}^{op} \to \mathcal{U}^{op}$ in **cat**_s which is the classifying discrete *fibration* in **cat**:

$$egin{aligned} &
u_{\mathcal{C}}\left(\mathcal{U}^{op}
ight)\left(m{c}
ight) = m{cat}_{s}\left(\mathcal{C}/m{c},\mathcal{U}^{op}
ight) \ &= m{cat}_{s}\left(\left(\mathcal{C}/m{c}
ight)^{op},\mathcal{U}
ight) \ &= \mathcal{U}_{\hat{\mathcal{C}}}\left(m{c}
ight) \end{aligned}$$

This suggests we can look for a 2-dimensional analogue.



Ssomething very similar appears in Weber's work on classifying discrete *opfibrations*. Weber considers a 2-adjunction for building the classifier in $Cat(\hat{C})$:

$$\mathsf{Cat} \xrightarrow[]{E_{\mathcal{C}}} \mathsf{Cat} \left(\hat{\mathcal{C}} \right)$$

Here, $\operatorname{Sp}_{C}(D)(c) = \operatorname{Cat}((C/c)^{op}, D)$, which looks strikingly familiar.



To properly compare them, we can factor Weber's construction through duality involutions to reveal a 2-dimensional nerve $\nu_C \colon \mathbf{Cat} \to \mathbf{Cat} \left(\hat{C} \right).$

$$\operatorname{Cat}_{\stackrel{op}{\underset{op}{\leftarrow}}}^{\stackrel{op}{\leftarrow}}\operatorname{Cat}^{co}_{\stackrel{co}{\underset{\nu_{co}}{\overset{co}{\leftarrow}}}}^{\int_{c}^{co}}\operatorname{Cat}\left(\hat{C}\right)^{co}_{\stackrel{co}{\underset{op}{\leftarrow}}}^{\stackrel{op}{\leftarrow}}\operatorname{Cat}\left(\hat{C}\right)$$

The Grothendieck construction \int_C is an **oplax colimit**. Its dual, used in the H-S construction, is a **lax colimit**. This suggests the H-S lifting functor is a form of **lax base change**, which is precisely what we will construct in the relative setting.



Our Contributions

We generalise the H-S construction in two ways:

- 1. From presheaves to fibrations.
- 2. From an absolute to a relative setting.



We move from the 2-category of presheaves of categories $Cat(\hat{C})$ to the 2-category of fibrations Fib_{C} .

Why?

The forgetful functor $Cat(\hat{C}) \simeq sFib_C \rightarrow Fib_C$ is not a bi-equivalence; it is thus helpful to work with the weak structure (fibrations) rather than strict structure (split fibrations with strictly-preserved splittings).



We relativise the lifting construction from the total category 2-functor \int_A : **Fib**_A \rightarrow **Cat** to an arbitrary fibration $p: A \rightarrow B$ (as \int_A is precisely postcomposition with $A \rightarrow 1$ under the identification **Cat** \cong **Fib**₁).

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- Our lifting is from $Fib_B \rightarrow Fib_A$.

Main Motivation: Iteration This allows us to **iterate** the H-S construction.



Why Relativise? The Need for Iteration

The Goal

If ordinary H-S lifting gives presheaf models of type theory, a relative version is what's needed for **internal presheaf models** within another model.

An internal category *D* in a model \hat{C} corresponds to a fibration $p_D: D \rightarrow C$. Our relative lifting takes a universe \mathcal{U}_C in **Fib**_{*C*} and lifts it along p_D to get a universe \mathcal{U}_D in **Fib**_{*D*}—a universe of internal presheaves on *D*.



Example: Cubical/Guarded Type Theory

Bizjak et al. (2016)

They described what amounts to the presheaf model of cubical type theory *internal* to the presheaf model of guarded dependent type theory.

Lacking the relative lifting tools developed here, they had to describe the guarded and cubical aspects simultaneously in a monolithic construction.

Our work provides the modular tools to build such models compositionally.

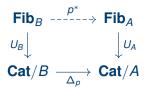


Finding the Right Adjunction

To generalise H-S lifting, we need a right pseudo-adjoint to a "sum-like" functor from Fib_A to Fib_B .



For any functor $p: A \rightarrow B$, the base change 2-functor $\Delta_p: \operatorname{Cat}/B \rightarrow \operatorname{Cat}/A$ restricts to a base change 2-functor $p^*: \operatorname{Fib}_B \rightarrow \operatorname{Fib}_A$ as follows:



The base change 2-functor p^* : **Fib**_{*B*} \rightarrow **Fib**_{*A*} always has a left pseudo-adjoint $p_!$: **Fib**_{*A*} \rightarrow **Fib**_{*B*} that deserves to be called the *sum* (of fibrations) along p, but this does *not* factor through the sum of Σ_p of displayed categories.



Instead, we consider the simple post-composition 2-functor Σ_p : **Cat**/ $A \rightarrow$ **Cat**/B.

Key Fact (Streicher)

When $p: A \rightarrow B$ is a **fibration**, this post-composition 2-functor restricts to fibrations:

$$\Sigma_p^{opl}$$
: Fib_A \rightarrow Fib_B

This functor is fibred in the sense of preserving cartesian arrows.

This is the "sum-like" 2-functor whose right pseudo-adjoint will give us the generalised H-S lifting.



Oplax Base Change

Main Result

The oplax sum 2-functor has a right **pseudo-adjoint**, which we call the **oplax base change** Δ_p^{opl} .

This gives us the central pseudo-adjunction of our work:

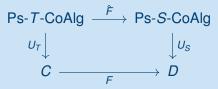
$$\mathsf{Fib}_{A} \xrightarrow{\Sigma_{P}^{opl}} \mathsf{Fib}_{B}$$



Existence of the Pseudo-Adjoint (Abstractly)

Theorem (Nunes, 2016)

Given a commutative diagram of pseudofunctors



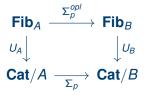
if *F* has a right pseudo-adjoint and Ps-*T*-CoAlg has descent objects, then \hat{F} has a right pseudo-adjoint.

(Thanks to Nathanael Arkor for this observation.)



Applying the Theorem to Our Case (1/2)

We apply this theorem to our diagram of oplax sums:



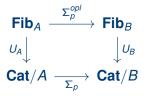
The conditions hold:

 Coalgebras: Fib_X is equivalent to the 2-category of pseudo-coalgebras for a comonad on Cat/X (Emmenegger et al., 2024).



Applying the Theorem to Our Case (1/2)

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The conditions hold:

- Coalgebras: Fib_X is equivalent to the 2-category of pseudo-coalgebras for a comonad on Cat/X (Emmenegger et al., 2024).
- Base Adjoint: The base functor Σ_p has a right pseudo-adjoint Δ_p (change of base).



Applying the Theorem to Our Case (2/2)

The conditions hold:

▶ **Descent:** Fib_A is bicomplete, so it has descent objects.

Conclusion

All conditions of Nunes's theorem are met, so Σ_{ρ}^{opl} must have a right pseudo-adjoint.



While the abstract proof guarantees existence, we also provide a constructive formula.

Formula for Oplax Base Change For a fibration $F \in \mathbf{Fib}_B$, the corresponding pseudofunctor for $\Delta_p^{opl}(F)$ over A is given by a fibred version of the Yoneda principle:

$$\Delta_{
ho}^{opl}(F)_{ullet}\cong \mathsf{Fib}_{B}\left(\Sigma_{
ho}^{opl}(\mathbf{y}_{A}(-)),F
ight)$$

where \mathbf{y}_A is the fibred Yoneda embedding.



The original H-S lifting has a contravariant aspect (e.g., \mathcal{U}^{op}). To capture this, we conjugate our adjunction with duality.

Lax Base Change

The lax base change Δ_{ρ}^{lax} : Fib_B \rightarrow Fib_A is defined as:

$$\Delta_p^{lax}(E) := \left(\Delta_p^{opl}(E^{op})\right)^{op}$$

Definition: Relative H-S Lifting

The relative Hofmann–Streicher lifting of a fibred category $E \in \mathbf{Fib}_B$ along a fibration $p: A \rightarrow B$ is the lax base change $\Delta_p^{lax}(E) \in \mathbf{Fib}_A$.



Sanity Check: Recovering the Classic Case

Does our generalisation recover the original? Yes.

The Setup

Let B = 1, so p is the unique fibration $!_A : A \rightarrow 1$. Then **Fib**₁ \cong **Cat**. Our lifting becomes a 2-functor:

 $\Delta_{!_{\mathcal{A}}}^{lax}$: **Cat** \rightarrow **Fib**_{\mathcal{A}}.



For a category $E \in Cat$, the fibres of the resulting fibration $\Delta_{l_A}^{lax}(E)$ over an object $a \in A$ are:

$$\Delta_{!_{A}}^{lax}(E)_{\bullet}(a) \cong \operatorname{Cat}\left(\Sigma_{!_{A}}^{opl}(\mathbf{y}_{A}(a)), E^{op}\right)^{op}$$
$$\cong \operatorname{Cat}\left(A/a, E^{op}\right)^{op}$$
$$\cong \operatorname{Cat}\left((A/a)^{op}, E\right)$$

~ ~

This is exactly the formula for the fibres of the classic H-S lifting.



Key Application: Iterated Lifting

Example

1. Start: A universe of sets $\mathcal{U} \in Cat \cong Fib_1$.



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- 2. Lift 1 (Classic): Lift \mathcal{U} along $!_C \colon C \to 1$ to get a universe of presheaves $\mathcal{U}_C \in \mathbf{Fib}_C$.

$$\mathcal{U}_{\mathcal{C}} := \Delta_{!_{\mathcal{C}}}^{lax}(\mathcal{U})$$



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3. Lift 2 (Relative): Lift U_C along an internal category fibration $p_D: D \rightarrow C$ to get a universe of internal presheaves on D.

$$\mathcal{U}_D := \Delta_{\rho_D}^{lax}(\mathcal{U}_C) \in \mathsf{Fib}_D$$



Summary of Contributions

- We defined a relative Hofmann-Streicher lifting, generalising the classic construction from an absolute setting (Cat → Fib_A) to a relative one (Fib_B → Fib_A) parameterised by a fibration.
- We showed this lifting arises as the right pseudo-adjoint (Δ^{lax}_p) in a new pseudo-adjunction on 2-categories of fibrations.
- This framework enables iterated lifting, which is a necessary tool for constructing internal models of advanced type theories.



Future Work & Connections

Concurrency Semantics

Our construction is a different generalisation of H-S lifting than the saturation monad of Fiore, Cattani & Winskel. It would be interesting to explore its applications in concurrency.

2-Dimensional Fibrations

The existence of these pseudo-adjoints for all base fibrations suggests a new kind of 2-dimensional bifibred structure, which merits further investigation.



Thank you!