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Hofmann–Streicher lifting of fibred categories

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Background: Universes & Models

Universes ($\mathcal{U}$)

- In set theory (Grothendieck): A tool for controlling sizes of categories.
- In type theory (Martin-Löf): A type of types.

By the mid 1990s, Martin-Löf type theory had acquired several credible notions of *categorical model*.

Motivating question

Are models of type theory stable under standard categorical constructions like forming presheaf categories?

Hofmann–Streicher Lifting (1997)

The H-S construction answers this for presheaves.

Given a universe $p\colon \tilde{\mathcal{U}} \to \mathcal{U}$ in **Set** and a $\mathcal{U}$-small category C, Hofmann and Streicher constructed a universe $p_{\hat{C}}\colon \tilde{\mathcal{U}}_{\hat{C}} \to \mathcal{U}_{\hat{C}}$ in the presheaf category $\hat{C}$.

The lifted universe is defined as follows, according to a reformulation by Awodey (2024):

$$\mathcal{U}_{\hat{C}}(c) = \mathbf{Cat}\left((C/c)^{op}, \mathcal{U}\right)$$
$$\tilde{\mathcal{U}}_{\hat{C}}(c) = \mathbf{Cat}\left((C/c)^{op}, \tilde{\mathcal{U}}\right)$$

H-S Lifting and Universal Properties

While universes aren't defined by a universal property, the H-S lifting construction *preserves* universal properties when they exist.

Example: Subobject Classifiers

The subobject classifier in **Set** is the map $\{1\} \hookrightarrow \mathbf{2}$.
Applying the Hofmann–Streicher lifting to **2** yields the subobject classifier $\Omega_{\hat{C}}$ in the presheaf topos $\hat{C}$:

$$\begin{aligned}\Omega_{\hat{C}}(c) &= \mathbf{Cat}\left((C/c)^{op}, \mathbf{2}\right) \\ &= \mathrm{Sieve}_C(c)\end{aligned}$$

A Functorial View (Awodey, 2024)

Awodey showed that the H-S lifting is the action of a functor, the **categorical nerve** ν_C.

Insight 1: An Adjunction

The nerve $\nu_C \colon \mathbf{cat}_s \to \hat{C}$ is the right adjoint to the category of elements functor $\int_C \colon \hat{C} \to \mathbf{cat}_s$.

$$\hat{C} \underset{\nu_C}{\overset{\int_C}{\rightleftarrows}} \mathbf{cat}_s \qquad (\int_C \dashv \nu_C)$$

It is defined as $\nu_C(D)(c) = \mathbf{cat}_s(C/c, D)$.

H-S Lifting via the Nerve

Insight 2: The Action on a Universe

The Hofmann–Streicher lifting of a universe $\tilde{\mathcal{U}} \to \mathcal{U}$ in **Set** to $\hat{C}$ is obtained by applying the categorical nerve to the arrow $\tilde{\mathcal{U}}^{op} \to \mathcal{U}^{op}$ in $\mathbf{cat}_s$ which is the classifying discrete *fibration* in **cat**:

$$\begin{aligned} \nu_C\left(\mathcal{U}^{op}\right)(c) &= \mathbf{cat}_s\left(C/c, \mathcal{U}^{op}\right) \\ &= \mathbf{cat}_s\left((C/c)^{op}, \mathcal{U}\right) \\ &= \mathcal{U}_{\hat{C}}(c) \end{aligned}$$

This suggests we can look for a 2-dimensional analogue.

A 2-Dimensional Analogue (Weber, 2007)

Ssomething very similar appears in Weber's work on classifying discrete *opfibrations*.

Weber considers a 2-adjunction for building the classifier in **Cat** $\left(\hat{C}\right)$:

$$\mathbf{Cat} \underset{\mathrm{Sp}_C}{\overset{E_C}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Cat}\left(\hat{C}\right)$$

Here, $\mathrm{Sp}_C(D)(c) = \mathbf{Cat}((C/c)^{op}, D)$, which looks strikingly familiar.

Connecting Awodey and Weber

To properly compare them, we can factor Weber's construction through duality involutions to reveal a 2-dimensional nerve $\nu_C \colon \mathbf{Cat} \to \mathbf{Cat}\left(\hat{C}\right)$.

$$\mathbf{Cat} \underset{\overset{\rightarrow}{op}}{\overset{\overset{op}{\leftarrow}}{\perp}} \mathbf{Cat}^{co} \underset{\overset{\rightarrow}{\nu_C^{co}}}{\overset{\overset{\int_C^{co}}{\leftarrow}}{\perp}} \mathbf{Cat}\left(\hat{C}\right)^{co} \underset{\overset{\rightarrow}{op}}{\overset{\overset{op}{\leftarrow}}{\perp}} \mathbf{Cat}\left(\hat{C}\right)$$

The Grothendieck construction $\int_C$ is an **oplax colimit**. Its dual, used in the H-S construction, is a **lax colimit**. This suggests the H-S lifting functor is a form of **lax base change**, which is precisely what we will construct in the relative setting.

Our Contributions

We generalise the H-S construction in two ways:

1. **From presheaves to fibrations.**
2. **From an absolute to a relative setting.**

Generalisation 1: From Presheaves to Fibrations

We move from the 2-category of presheaves of categories $\mathbf{Cat}(\hat{C})$ to the 2-category of fibrations $\mathbf{Fib}_C$.

Why?

The forgetful functor $\mathbf{Cat}\left(\hat{C}\right) \simeq \mathbf{sFib}_C \to \mathbf{Fib}_C$ is not a bi-equivalence; it is thus helpful to work with the weak structure (fibrations) rather than strict structure (split fibrations with strictly-preserved splittings).

Generalisation 2: From Absolute to Relative

We relativise the lifting construction from the total category 2-functor $\int_A \colon \mathbf{Fib}_A \to \mathbf{Cat}$ to an arbitrary fibration $p \colon A \rightarrow B$ (as $\int_A$ is precisely postcomposition with $A \to \mathbf{1}$ under the identification $\mathbf{Cat} \cong \mathbf{Fib}_{\mathbf{1}}$).

- The classic lifting is from $\mathbf{Cat} \to \mathbf{Fib}_A$.

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Main Motivation: Iteration

This allows us to **iterate** the H-S construction.

Why Relativise? The Need for Iteration

The Goal

If ordinary H-S lifting gives presheaf models of type theory, a relative version is what's needed for **internal presheaf models** within another model.

An internal category D in a model $\hat{C}$ corresponds to a fibration $p_D \colon D \twoheadrightarrow C$.

Our relative lifting takes a universe $\mathcal{U}_C$ in $\mathbf{Fib}_C$ and lifts it along p_D to get a universe $\mathcal{U}_D$ in $\mathbf{Fib}_D$—a universe of internal presheaves on D.

Example: Cubical/Guarded Type Theory

Bizjak et al. (2016)

They described what amounts to the presheaf model of cubical type theory *internal* to the presheaf model of guarded dependent type theory.
Lacking the relative lifting tools developed here, they had to describe the guarded and cubical aspects simultaneously in a monolithic construction.
Our work provides the modular tools to build such models compositionally.

Finding the Right Adjunction

To generalise H-S lifting, we need a right pseudo-adjoint to a "sum-like" functor from $\mathbf{Fib}_A$ to $\mathbf{Fib}_B$.

The Standard "Sum of Fibrations" ($p_!$)

For any functor $p\colon A \to B$, the base change 2-functor $\Delta_p\colon \mathbf{Cat}/B \to \mathbf{Cat}/A$ restricts to a base change 2-functor $p^*\colon \mathbf{Fib}_B \to \mathbf{Fib}_A$ as follows:

$$
\begin{array}{ccc}
\mathbf{Fib}_B & \overset{p^*}{\dashrightarrow} & \mathbf{Fib}_A \\
{\scriptstyle U_B}\downarrow & & \downarrow{\scriptstyle U_A} \\
\mathbf{Cat}/B & \underset{\Delta_p}{\longrightarrow} & \mathbf{Cat}/A
\end{array}
$$

The base change 2-functor $p^*\colon \mathbf{Fib}_B \to \mathbf{Fib}_A$ always has a left pseudo-adjoint $p_!\colon \mathbf{Fib}_A \to \mathbf{Fib}_B$ that deserves to be called the *sum (of fibrations) along* p, but this does *not* factor through the sum of Σ_p of displayed categories.

The Oplax Sum (Σ_p^{opl})

Instead, we consider the simple post-composition 2-functor $\Sigma_p\colon \mathbf{Cat}/A \to \mathbf{Cat}/B$.

Key Fact (Streicher)

When $p\colon A \twoheadrightarrow B$ is a **fibration**, this post-composition 2-functor restricts to fibrations:

$$\Sigma_p^{opl}\colon \mathbf{Fib}_A \to \mathbf{Fib}_B$$

This functor is fibred in the sense of preserving cartesian arrows.

This is the "sum-like" 2-functor whose right pseudo-adjoint will give us the generalised H-S lifting.

Oplax Base Change

Main Result

The oplax sum 2-functor has a right **pseudo-adjoint**, which we call the **oplax base change** Δ_p^{opl}.

This gives us the central pseudo-adjunction of our work:

$$\mathbf{Fib}_A \underset{\Delta_p^{opl}}{\overset{\Sigma_p^{opl}}{\rightleftarrows}} \mathbf{Fib}_B \qquad (\Sigma_p^{opl} \dashv \Delta_p^{opl})$$

Existence of the Pseudo-Adjoint (Abstractly)

Theorem (Nunes, 2016)

Given a commutative diagram of pseudofunctors

$$\begin{array}{ccc} \text{Ps-}T\text{-CoAlg} & \xrightarrow{\hat{F}} & \text{Ps-}S\text{-CoAlg} \\ {\scriptstyle U_T}\downarrow & & \downarrow{\scriptstyle U_S} \\ C & \xrightarrow[F]{} & D \end{array}$$

if F has a right pseudo-adjoint and Ps-T-CoAlg has descent objects, then $\hat{F}$ has a right pseudo-adjoint.

(Thanks to Nathanael Arkor for this observation.)

Applying the Theorem to Our Case (1/2)

We apply this theorem to our diagram of oplax sums:

$$\begin{array}{ccc} \mathbf{Fib}_A & \xrightarrow{\Sigma_p^{opl}} & \mathbf{Fib}_B \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle U_B} \\ \mathbf{Cat}/A & \xrightarrow[\Sigma_p]{} & \mathbf{Cat}/B \end{array}$$

The conditions hold:

- **Coalgebras:** $\mathbf{Fib}_X$ is equivalent to the 2-category of pseudo-coalgebras for a comonad on $\mathbf{Cat}/X$ (Emmenegger et al., 2024).

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The conditions hold:

- **Coalgebras:** $\mathbf{Fib}_X$ is equivalent to the 2-category of pseudo-coalgebras for a comonad on $\mathbf{Cat}/X$ (Emmenegger et al., 2024).
- **Base Adjoint:** The base functor Σ_p has a right pseudo-adjoint Δ_p (change of base).

Applying the Theorem to Our Case (2/2)

The conditions hold:

- **Descent: $\mathbf{Fib}_A$** is bicomplete, so it has descent objects.

Conclusion

All conditions of Nunes's theorem are met, so Σ_p^{opl} must have a right pseudo-adjoint.

An Explicit Formula for the Adjoint

While the abstract proof guarantees existence, we also provide a constructive formula.

Formula for Oplax Base Change

For a fibration $F \in \mathbf{Fib}_B$, the corresponding pseudofunctor for $\Delta_p^{opl}(F)$ over A is given by a fibred version of the Yoneda principle:

$$\Delta_p^{opl}(F)_\bullet \cong \mathbf{Fib}_B\left(\Sigma_p^{opl}(\mathbf{y}_A(-)), F\right)$$

where $\mathbf{y}_A$ is the fibred Yoneda embedding.

Defining the Relative H-S Lifting

The original H-S lifting has a contravariant aspect (e.g., $\mathcal{U}^{op}$). To capture this, we conjugate our adjunction with duality.

Lax Base Change

The **lax base change** $\Delta_p^{lax}\colon \mathbf{Fib}_B \to \mathbf{Fib}_A$ is defined as:

$$\Delta_p^{lax}(E) := \left(\Delta_p^{opl}(E^{op})\right)^{op}$$

Definition: Relative H-S Lifting

The relative Hofmann–Streicher lifting of a fibred category $E \in \mathbf{Fib}_B$ along a fibration $p\colon A \twoheadrightarrow B$ is the lax base change $\Delta_p^{lax}(E) \in \mathbf{Fib}_A$.

Sanity Check: Recovering the Classic Case

Does our generalisation recover the original? **Yes.**

The Setup

Let $B = \mathbf{1}$, so p is the unique fibration $!_A \colon A \twoheadrightarrow \mathbf{1}$. Then $\mathbf{Fib_1} \cong \mathbf{Cat}$.
Our lifting becomes a 2-functor:

$$\Delta^{lax}_{!_A} \colon \mathbf{Cat} \to \mathbf{Fib}_A.$$

Sanity Check: The Calculation

For a category $E \in \mathbf{Cat}$, the fibres of the resulting fibration $\Delta_{!_A}^{lax}(E)$ over an object $a \in A$ are:

$$\begin{aligned}\Delta_{!_A}^{lax}(E)_{\bullet}(a) &\cong \mathbf{Cat}\left(\Sigma_{!_A}^{opl}(\mathbf{y}_A(a)), E^{op}\right)^{op} \\ &\cong \mathbf{Cat}(A/a, E^{op})^{op} \\ &\cong \mathbf{Cat}\left((A/a)^{op}, E\right)\end{aligned}$$

This is exactly the formula for the fibres of the classic H-S lifting.

Key Application: Iterated Lifting

Example

1. **Start:** A universe of sets $\mathcal{U} \in \mathbf{Cat} \cong \mathbf{Fib_1}$.

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2. **Lift 1 (Classic):** Lift $\mathcal{U}$ along $!_C \colon C \to \mathbf{1}$ to get a universe of presheaves $\mathcal{U}_C \in \mathbf{Fib}_C$.

$$\mathcal{U}_C := \Delta^{lax}_{!_C}(\mathcal{U})$$

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$$\mathcal{U}_C := \Delta^{lax}_{!_C}(\mathcal{U})$$

3. **Lift 2 (Relative):** Lift $\mathcal{U}_C$ along an internal category fibration $p_D \colon D \twoheadrightarrow C$ to get a universe of internal presheaves on D.

$$\mathcal{U}_D := \Delta^{lax}_{p_D}(\mathcal{U}_C) \in \mathbf{Fib}_D$$

Summary of Contributions

- We defined a **relative** Hofmann-Streicher lifting, generalising the classic construction from an absolute setting ($\mathbf{Cat} \to \mathbf{Fib}_A$) to a relative one ($\mathbf{Fib}_B \to \mathbf{Fib}_A$) parameterised by a fibration.
- We showed this lifting arises as the right **pseudo-adjoint** (Δ_p^{lax}) in a new pseudo-adjunction on 2-categories of fibrations.
- This framework enables **iterated** lifting, which is a necessary tool for constructing internal models of advanced type theories.

Future Work & Connections

Concurrency Semantics

Our construction is a different generalisation of H-S lifting than the saturation monad of Fiore, Cattani & Winskel. It would be interesting to explore its applications in concurrency.

2-Dimensional Fibrations

The existence of these pseudo-adjoints for all base fibrations suggests a new kind of 2-dimensional bifibred structure, which merits further investigation.

Thank you!