Bicategories of algebras for relative pseudomonads

Nathanael Arkor
Tallinn University of Technology
n@arkor.co

Philip Saville
University of Oxford
philip.saville@cs.ox.ac.uk

Andrew Slattery
University of Leeds
mmawsl@leeds.ac.uk

1 Relative (pseudo)monads & the presheaf construction

Definition 1.1 (Relative monad [1])**.** Let $J \colon \mathbb{C} \to \mathbb{D}$ be a functor between categories. A *J-relative monad* $(T, i, (-)^*)$ comprises

- *units* $i_X \colon JX \to TX$ in $\mathbb{D}$ for $X \in \mathrm{ob}\,\mathbb{C}$;
- *extensions* $f^* \colon TX \to TY$ in $\mathbb{D}$ for $f \colon JX \to TY$,

satisfying $f = f^* i$, $(f^* g)^* = f^* g^*$, $i_X^* = 1_{TX}$ for $f \colon JX \to TY$ and $g \colon JW \to TX$.

If J is instead a pseudofunctor between *bicategories*, we may weaken the above equalities to invertible 2-cells, obtaining a *relative pseudomonad* [2].

Example 1.2. (Presheaves) The presheaf construction $P \colon \mathrm{Cat} \to \mathrm{CAT}$ has the structure of a relative pseudomonad. The units are Yoneda embeddings $y_X \colon X \to PX$, and the extension of $f \colon X \to PY$ is given by its left extension along y_X (i.e. by taking a certain colimit):

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[dr, "f"'] & PX \arrow[d, dashed, "f^* := y_X \triangleright f"] \arrow[dl, phantom, "\cong"] \\ & PY \end{tikzcd}$$

2 The bicategory of pseudoalgebras for a relative pseudomonad

The Kleisli bicategory for a relative pseudomonad was constructed in [2]; we construct the bicategory of pseudoalgebras.

Definition 2.1 (T-pseudoalgebra)**.** For a J-relative pseudomonad T, a *T-pseudoalgebra* $(A, (-)^a; \tilde{a}, \hat{a})$ comprises

- an *underlying object* A in $\mathrm{ob}\,\mathbb{D}$;
- *algebra extensions* $f^a \colon TX \to A$ in $\mathbb{D}$ for $f \colon JX \to A$,

along with two families of invertible 2-cells $\tilde{a}_f \colon f \to f^a i_X$ and $\hat{a} \colon (f^a g)^a \to f^a g^*$ for $f \colon JX \to A$ and $g \colon JW \to TX$.

When J is the identity, this definition reduces to that of a pseudoalgebra for a *no-iteration pseudomonad* as defined in [3].

Defining notions of algebra morphism and algebra transformation, we may assemble pseudoalgebras into a bicategory.

Theorem 2.2. *Let T be a relative pseudomonad. T-pseudoalgebras, pseudomorphisms and transformations form a bicategory* $\mathrm{PsAlg}(T)$*, called the* bicategory of T-pseudoalgebras.

To demonstrate that our construction is correct, we find that $\mathrm{PsAlg}(T)$ satisfies the following universal property.

Theorem 2.3. *Let T be a J-relative pseudomonad. The bicategory of pseudoalgebras and pseudomorphisms forms a relative pseudoadjunction which is bi-terminal among resolutions of T (the J-relative pseudoadjunctions that induce T).*

$$\begin{tikzcd} & \mathrm{PsAlg}(T) \arrow[dr, "U_T"] \arrow[d, phantom, "\dashv"] & \\ \mathbb{C} \arrow[ur, "F_T"] \arrow[rr, "J"'] & {} & \mathbb{D} \end{tikzcd}$$

Likewise, the Kleisli resolution is bi-initial, and there is an canonical fully faithful pseudofunctor $I_T \colon \mathrm{Kl}(T) \to \mathrm{PsAlg}(T)$ which is the embedding of the free T-pseudoalgebras.

3 Idempotence does not imply algebraic idempotence

We explore two notions of idempotence for relative monads.

Definition 3.1. (Idempotent relative monad) A relative monad $(T, i, (-)^*)$ is *idempotent* if the extension operator $(-)^* \colon \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$ is inverse to precomposition with the unit $- \circ i_X \colon \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$, for all X, Y in $\mathrm{ob}\,\mathbb{C}$.

Definition 3.2. (Algebraically idempotent relative monad) A relative monad $(T, i, (-)^*)$ is *algebraically idempotent* if the algebra extension $(-)^a \colon \mathbb{D}(JX, A) \to \mathbb{D}(TX, A)$ is inverse to precomposition with the unit $- \circ i_X \colon \mathbb{D}(JX, A) \to \mathbb{D}(TX, A)$, for all X in $\mathrm{ob}\,\mathbb{C}$ and $(A, (-)^a)$ in $\mathrm{Alg}(T)$.

Algebraic idempotence implies idempotence (which, by definition, is algebraic idempotence restricted to free algebras). In fact, in the special case where J is an identity, the two notions coincide. However, in general they do not. A counterexample is given by the following small category $\mathbb{D}$,

$$\begin{tikzcd} J \arrow[r, "i"] & T \arrow[r, bend left, "h"] \arrow[r, bend right, "g"'] & A \end{tikzcd}$$

subject to $gi = hi$. Considering J and T as functors from the point to $\mathbb{D}$, T is a J-relative monad which is idempotent but not algebraically idempotent.

4 The presheaf construction *is* algebraically lax-idempotent

In the bicategorical setting, we may generalise from (algebraic) idempotence to *(algebraic) lax-idempotence*, asking only for an adjunction $(-)^* \dashv - \circ i_X$ or $(-)^a \dashv - \circ i_X$. The presheaf relative pseudomonad P is lax-idempotent by construction; however, showing it to be algebraically lax-idempotent is (perhaps surprisingly) nontrivial.

Proposition 4.1. *Let $(A, (-)^a)$ be a P-pseudoalgebra. Then $A \in \mathrm{CAT}$ is a cocomplete category.*

The colimit of a diagram $f \colon D \to A$ can be computed as the image of the terminal presheaf under $f^a \colon PD \to A$.

We can also show that every pseudomorphism of algebras $(A, (-)^a) \to (B, (-)^b)$ is cocontinuous as a functor $A \to B$. Consequently, we have the following.

Theorem 4.2. *The presheaf relative pseudomonad P is algebraically lax-idempotent.*

Corollary 4.3. *The bicategory* Dist *of small categories and distributors is biequivalent to the full sub-2-category of the 2-category of locally small cocomplete categories and cocontinuous functors, spanned by the presheaf categories.*

References

[1] Thomas Altenkirch, James Chapman, and Tarmo Uustalu. Monads need not be endofunctors. *Logical Methods in Computer Science*, 11(1), 2015.

[2] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. *Selecta Mathematica*, 24(3):2791—-2830, 2018.

[3] Francisco Marmolejo and Richard J Wood. No-iteration pseudomonads. *Theory and Applications of Categories*, 28(14):371–402, 2013.