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Presheaf Algebras are Cocomplete Categories

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Outline

Monads

Definition

A *monad in Kleisli presentation* $(T, i, ^*)$ on a category $\mathbb{C}$ comprises

- for each object X in $\mathbb{C}$, an object TX in $\mathbb{C}$ and *unit map* $i_X : X \to TX$,
- for every map $f : X \to TX$ its *extension* $f^* : TX \to TX$,
  satisfying the following three equations for all $f : Y \to TZ$, $g : X \to TY$:

$$
\begin{aligned}
f &= f^* i_Y, \\
(f^* g)^* &= f^* g^*, \\
i_X * &= 1_{TX}.
\end{aligned}
$$

This presentation is equivalent to the usual one, in that each structure induces the other.

Relative monads

However, the definition of extension system does not reference iteration of the action of T, and so it can be more easily generalised to the notion of a monad *along* some base functor $J : \mathbb{C} \to \mathbb{D}$.

Definition

(Relative monad, Altenkirch et al. 2014) A *relative monad* $(T, i, {}^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object X in $\mathbb{C}$, an object TX in $\mathbb{D}$ and *unit map* $i_X : JX \to TX$,
- for every map $f : JX \to TY$ an *extension* $f^* : TX \to TY$,
  satisfying the following three equations for all $f : JY \to TZ$, $g : JX \to TY$:

$$f = f^* i_Y,$$
$$(f^* g)^* = f^* g^*,$$
$$i_X * = 1_{TX}.$$

This is identical to the previous slide's definition up to the use of J to ensure objects lie in the required category.

Relative pseudomonads

We can categorify this definition, considering now 2-categories $\mathbb{C}$ and $\mathbb{D}$.

Definition

(Relative pseudomonad, Fiore et al. 2018) A *relative pseudomonad* $(T, i, {}^*; \eta, \mu, \theta)$ along a 2-functor $J : \mathbb{C} \to \mathbb{D}$ comprises

- for each object X in $\mathbb{C}$, an object TX in $\mathbb{D}$ and *unit map* $i_X : JX \to TX$,
- for every X, Y an extension functor between hom-categories $\mathbb{D}(JX, TY) \xrightarrow{(-)^*} \mathbb{D}(TX, TY)$,

along with three invertible families of 2-cells:

- $\eta_f : f \to f^* i_Y$ for $f : JY \to TZ$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JY \to TZ$, $g : JX \to TY$, and
- $\theta_X : i_X^* \to 1_{TX}$ for X in $\mathbb{C}$.

satisfying two coherence diagrams.

Relative pseudomonads

These two coherence diagrams are:

$$\begin{array}{ccccc} ((f^*g)^*h)^* & & \xrightarrow{\mu} & & (f^*g)^*h^* \\ {\scriptstyle\mu}\downarrow & & & & \downarrow{\scriptstyle\mu} \\ (f^*g^*h)^* & \xrightarrow[\mu]{} & f^*(g^*h)^* & \xrightarrow[\mu]{} & f^*g^*h^* \end{array}$$

$$\begin{array}{ccccc} f^* & \xrightarrow{\eta} & (f^*i)^* & \xrightarrow{\mu} & f^*i^* \\ & \searrow^{=} & & & \downarrow{\scriptstyle\theta} \\ & & & & f^*1 \end{array}$$

namely an associativity condition for μ and a unitality equation relating μ to the units η and θ.

Relative pseudomonads

Lemma

(Fiore et al. 2018) A relative pseudomonad furthermore satisfies the following three coherence conditions:

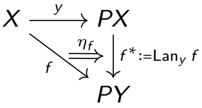
$$\begin{array}{ccc} f^*g & \xrightarrow{\eta} & (f^*g)^*i \\ & \searrow^{\eta} & \downarrow{\scriptstyle \mu} \\ & & f^*g^*i \end{array} \qquad \begin{array}{ccc} (i^*f)^* & \xrightarrow{\mu} & i^*f^* \\ & \searrow^{\theta} & \downarrow{\scriptstyle \theta} \\ & & f^* \end{array} \qquad \begin{array}{ccc} i & \xrightarrow{\eta} & i^*i \\ & \searrow^{=} & \downarrow{\scriptstyle \theta} \\ & & i \end{array}$$

The proof of this is analogous to showing the five original coherence axioms for a monoidal category follow from the pentagon and triangle axioms.

Example: The presheaf relative pseudomonad

The presheaf construction $X \mapsto PX$ cannot be given the structure of a pseudomonad, since it is not an endofunctor (due to size issues). However, it can be given the structure of a relative pseudomonad along the inclusion $J : \mathrm{Cat} \to \mathrm{CAT}$ as follows:

- the unit $y_X : X \to PX$ is given by the Yoneda embedding,
- the extension of a functor $f : X \to PY$ is given by the left Kan extension of f along the Yoneda embedding

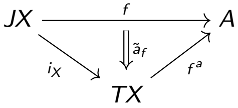


  which also defines the 2-cells $\eta_f : f \to f^* y$.
- the 2-cells $\mu_{f,g}$ and θ_X are defined by the universal property of the left Kan extension.

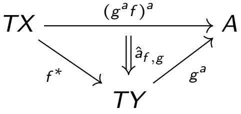
Pseudoalgebras

A *pseudoalgebra* for a J-relative pseudomonad $T : \mathbb{C} \to \mathbb{D}$ (or simply a T-pseudoalgebra) comprises

- an object $A \in \mathbb{D}$;
- a family of functors $(-)^a_X : \mathbb{D}[JX, A] \to \mathbb{D}[TX, A]$ for $X \in \mathbb{C}$;
- a natural family of invertible 2-cells $\tilde{a}_f : f \to f^a i_X$ for $f : JX \to A$ in $\mathbb{C}$:



- a natural family of invertible 2-cells $\hat{a}_{f,g} : (g^a f)^a \to g^a f^*$ for $f : JX \to TY$ and $g : JY \to A$ in $\mathbb{D}$:



satisfying two coherence equations.

Pseudoalgebras

These two coherence diagrams are:

$$
\begin{array}{ccccc}
((f^a g)^a h)^* & & \xrightarrow{\hat{a}} & & (f^a g)^a h^* \\
\downarrow \hat{a} & & & & \downarrow \hat{a} \\
(f^a g^* h)^a & \xrightarrow[\hat{a}]{} & f^a (g^* h)^* & \xrightarrow[\mu]{} & f^a g^* h^*
\end{array}
$$

$$
\begin{array}{ccccc}
f^a & \xrightarrow{\tilde{a}} & (f^a i)^a & \xrightarrow{\hat{a}} & f^a i^* \\
 & \searrow^{=} & & & \downarrow \theta \\
 & & & & f^a 1
\end{array}
$$

which resemble very closely the two coherence diagrams for a relative pseudomonad. We can make the precise with the notion of a free pseudoalgebra.

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Free pseudoalgebras

For every object Y in $\mathbb{C}$, there is a canonical pseudoalgebra structure on TY. The algebra extension operation is given by the extension functor

$$\mathbb{D}(JX, TY) \xrightarrow{(-)^*} \mathbb{D}(TX, TY),$$

while the families of 2-cells are given by the η and μ respectively—the required two coherence conditions are then given exactly by a pseudomonad's coherence conditions.

Further coherence

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Lemma

For a T-pseudoalgebra A the following diagram

$$\begin{array}{ccc} f^a g & \xrightarrow{\tilde{a}} & (f^a g)^a i \\ & \searrow^{\eta} & \downarrow{\scriptstyle \hat{a}} \\ & & g^a f^* i \end{array}$$

also commutes.

The proof is identical in form to the first of the three conditions in the previous lemma.

Morphisms

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We define notions of morphism between pseudoalgebras.

Definition

A *lax morphism* from T-pseudoalgebra $(A, {}^{a}; \tilde{a}, \hat{a})$ to $(B, {}^{b}; \tilde{b}, \hat{b})$ comprises a morphism $f : A \to B$ in $\mathbb{D}$ along with a transformation

$$\begin{array}{ccc} \mathbb{D}[JX, A] & \xrightarrow{f-} & \mathbb{D}[JX, B] \\ {\scriptstyle (-)^{a}} \downarrow & \overset{\bar{f}}{\swarrow} & \downarrow {\scriptstyle (-)^{b}} \\ \mathbb{D}[TX, A] & \xrightarrow[f-]{} & \mathbb{D}[TX, B] \end{array}$$

which amounts to a family of 2-cells

$$\bar{f}_g : (fg)^b \to fg^a$$

satisfying two coherence conditions.

If $\bar{f}$ is invertible, we say $(h, \bar{f})$ is a *pseudomorphism*, and if it is an identity, we say $(f, \bar{f})$ is a *strict morphism*.

Example

Given a T-pseudoalgebra $(A, {}^a; \tilde{a}, \hat{a})$, for any $X \in \mathbb{C}$ and $f : JX \to A$ the map $f^a : TX \to A$ has a pseudomorphism structure given by

$$\overline{f^a}_g = \hat{a}_{f,g} : (f^a g)^a \to f^a g^*.$$

In this case, the two pseudomorphism coherence conditions become precisely the two coherence conditions for a pseudoalgebra.

2-cells and Alg(T)

An algebra 2-cell is a 2-cell $f \to g$ such that

$$\begin{array}{ccc} (fg)^b & \xrightarrow{\alpha} & (kg)^b \\ {\scriptstyle \bar{f}_g}\big\downarrow & & \big\downarrow{\scriptstyle \bar{k}_g} \\ fg^a & \xrightarrow[\alpha]{} & kg^a \end{array}$$

commutes.

Pseudoalgebras, pseudomorphisms and algebra 2-cells form a 2-category Alg(T) (with lax or strict morphisms we denote the resulting 2-categories $\mathrm{Alg}_l(T)$ and $\mathrm{Alg}_s(T)$ respectively).

The small-presheaf pseudomonad

It is possible the sidestep size issues with the presheaf construction differently. On CAT we can define the small-presheaf pseudomonad P_s, which takes a locally-small category X to the locally-small category $P_s X$ of small presheaves (those which are small colimits of representable presheaves). Note that upon composition with the inclusion $J : \mathrm{Cat} \to \mathrm{CAT}$, P_s coincides with the unrestricted presheaf construction $P : \mathrm{Cat} \to \mathrm{CAT}$.

Small-presheaf pseudoalgebras

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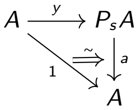
Characterising pseudoalgebras for P_s is simple: if (A, a) is a P_s-pseudoalgebra, the pseudoalgebra diagram



exhibits A as a reflective subcategory of the cocomplete category P_sA, and so A is cocomplete. Conversely, a cocomplete category A has a P_s-pseudoalgebra structure given by

$$P_sA \to A : \operatorname{colim} yf \mapsto \operatorname{colim} f$$

for $f : D \to A$ a small diagram in A.

The problem

One might therefore expect it to be similarly simple to prove that the P-pseudoalgebras are also small-cocomplete categories. However, we completely lack an analogue for the object $P_s A$ in the relative setting, so such a proof is impossible.

Furthermore, at the heart of the previous proof is the following fact.

Proposition

(Kock, 1995) Let (T, i, m) be a pseudomonad. Suppose that we have a reflective adjunction

$$m \dashv i.$$

Then for any T-pseudoalgebra (A, a) we also have a reflective adjunction

$$a \dashv i.$$

We will say that in the ordinary setting, T is lax-idempotent if and only if it is algebraically lax-idempotent.

Lax idempotency

We define two notions of lax idempotency for relative pseudomonads.

Definition

(Fiore et al.) A relative pseudomonad $(T, i, ^*)$ is *lax-idempotent* if we have an adjunction

$$(-)^* \dashv - \circ i$$

for which η is the unit.

Definition

A relative pseudomonad $(T, i, ^*)$ is *algebraically lax-idempotent* if for every T-pseudoalgebra we have an adjunction

$$(-)^a \dashv - \circ i$$

for which $\tilde{a}$ is the unit.

The one-dimensional case

In one dimension, to be idempotent is for $(-)^*$ to be a bijection, while to be algebraically idempotent is for $(-)^a$ to be a bijection for every algebra $(A, {}^a)$.

Proposition

In the relative setting, idempotent does not imply algebraically idempotent.

Since every 1-category is a 2-category, this will immediately imply that not every lax-idempotent relative pseudomonad is algebraically lax-idempotent.

Idempotent $\not\Longrightarrow$ algebraically idempotent

Proof.

Consider the three-object category $\mathbb{C}$ generated by the diagram

$$\begin{array}{ccc} J & \xrightarrow{\ i\ } & T \\ & & {\scriptstyle f}\downarrow\ \downarrow{\scriptstyle g} \\ & & A \end{array}$$

subject to $fi = gi := h$. We have functors $J, T : 1 \to \mathbb{C}$ and T is a relative monad with unit i and extension defined by $i^* = 1$. Then T is idempotent since $|[J, T]| = |[T, T]| = 1$.

However, A has two T-algebra structures sending h to f and g respectively , and yet $|[J, A]| = 1 \neq 2 = |[T, A]|$. Hence T is not algebraically idempotent. $\square$

Presheaf pseudoalgebras

Hence, although we have reflective adjunctions

$$(-)^* \dashv - \circ y$$

for the presheaf relative pseudomonad, we *cannot* yet conclude that for every presheaf pseudoalgebra $(A, {}^a; \tilde{a}, \hat{a})$ we have a reflective adjunction

$$(-)^a \dashv - \circ y,$$

or equivalently that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;y\;} & PX \\ & {}_{f}\searrow \;\overset{\tilde{a}_f}{\Longrightarrow} & \downarrow{\scriptstyle f^a} \\ & & A \end{array}$$

exhibits f^a as the left Kan extension of f along y. We will have to approach P-pseudoalgebras differently.

Presheaf pseudoalgebras

In Fiore et al. (2018) it was established that the Kleisli bicategory of P is the bicategory Prof of profunctors between small categories. We now give an explicit characterisation of the Eilenberg-Moore 2-category.

Theorem

The 2-category $\mathrm{Alg}(P)$ *is biequivalent to the 2-category of cocomplete categories, cocontinuous functors and natural transformations.*

Cocomplete $\implies$ algebra

Lemma

Let $D, A \in \mathrm{CAT}$ with D small and A cocomplete, let $f : D \to A$ and $g : PD \to A$ be functors with g cocontinuous, and let $\alpha : f \to gy$ be an invertible natural transformation. Then its transpose

$$\alpha^\sharp : \mathrm{Lan}_y f \to g$$

is also invertible.

Cocomplete $\implies$ algebra

Proposition

Suppose $A \in \mathrm{CAT}$ is cocomplete. Then A has the structure of a P-pseudoalgebra.

Proof.

For $f : D \to A$ define $f^a : PD \to A$ to be the left Kan extension of f along y (which exists because A is cocomplete). Then we have a family of isomorphisms $\tilde{a}_f : f \to f^a y$ given by the 2-cell part of the left Kan extension. Define the other part of the pseudoalgebra structure $\hat{a}_{f,g} : (f^a g)^a \to f^a g^*$ to be the transpose of the invertible transformation

$$f^a g \xrightarrow{f^a \eta_g} f^a g^* y,$$

noting by the previous lemma that $\hat{a}_{f,g}$ is invertible.
One may check via transposes that the coherence conditions are satisfied. □

Cocontinuous $\implies$ pseudomorphism

Proposition

Let $f : A \to B$ be a cocontinuous functor between cocomplete categories. Giving A and B the P-pseudoalgebra structures as above, f becomes a pseudomorphism of algebras.

Proof.

The 2-cell $\bar{f}_g : (fg)^b \to fg^a$ is define to be the transpose of the invertible 2-cell

$$f\tilde{a}_g : fg \to fg^a y,$$

noting by the same lemma as before that $\bar{f}_g$ is invertible.
One may check via transposes that the coherence conditions are satisfied. □

The inclusion of COC into Alg(P) is thus relatively straightforward. The other direction will take more work.

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Algebra $\implies$ cocomplete

Proposition

Let $(A, {}^a, \tilde{a}, \hat{a})$ be a P-pseudoalgebra. Then its underlying object $A \in \mathrm{CAT}$ is cocomplete; for a small diagram $f : D \to A$ we have

$$\operatorname{colim} f \cong f^a \operatorname{colim} y_D.$$

Consider the presheaf $s := \operatorname{colim} y_D \in PD$; it is the terminal presheaf sending every object of D to a singleton, and it has inclusion maps $\upsilon_d : yd \hookrightarrow s$ for all d. Our proof has the following structure:

1. We show that $f^a s$ is the apex of a cocone c under f.
2. We characterise cocones g under f.
3. Given a cocone g under f, we construct a map of cocones $z_g : f^a s \to gt$.
4. We characterise maps of cocones from c to g.
5. We show that there is a unique such cocone for any g.

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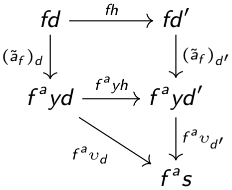
(1) The cocone c under f

The composites

$$fd \xrightarrow{(\tilde{a}_f)_d} f^a yd \xrightarrow{f^a \upsilon_d} f^a s$$

form a cocone under f with apex $f^a s$; indeed, for any $h \colon d \to d'$ the diagram



comprises a naturality square and the image of the colim y cocone under f^a, and thus commutes.

(2) Cocones under f

Define a small category D^t as follows:

$$\text{ob}\, D^t = \text{ob}\, D \sqcup \{t\},$$

$$\text{mor}\, D^t = \text{mor}\, D \sqcup \{1_t\} \sqcup \{d \xrightarrow{!_d} t : d \in \text{ob}\, D\},$$

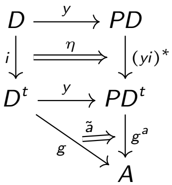
so that D^t is the category formed by freely adjoining a single terminal object to D. Then we have an inclusion $i : D \to D^t$, and a correspondence between

- functors $g : D^t \to A$ such that $gi = f$, and
- cocones under f.

The purpose of this correspondence is to be able to manipulate cocones under f using the pseudoalgebra structure.

(3) The map z_g

Let $g : D^t \to A$ be such a functor; we need to construct a map $z_g : f^a s \to gt$. In the diagram below



consider the objects $s \in PD$ and $yt \in PD^t$. Since t is terminal in D^t, yt is terminal in PD^t. The presheaf $(yi)^* s$ is in PD^t and by terminality we have a unique map $(yi)^* s \xrightarrow{!} yt$ in $P\mathbb{D}^t$.

Applying the functor g^a to this map, we obtain $g^a (yi)^* s \xrightarrow{g^a !} g^a yt$, with which we can form the composite

$$f^a s = (gi)^a s \xrightarrow{\tilde{a}} (g^a yi)^a s \xrightarrow{\hat{a}} g^a (yi)^* s \xrightarrow{g^a !} g^a yt \xrightarrow{\tilde{a}^{-1}} gt.$$

This gives us a map $f^a s \to gt$, and this is how we define the desired z_g.

(2, cont.) The map z_g is a map of cocones

We need to show that the diagram

$$
\begin{array}{ccccc}
fd & = & gid & & \\
\Big\downarrow{\scriptstyle (\tilde{a}_f)_d} & & & \searrow{\scriptstyle g!} & \\
f^a yd & \xrightarrow[f^a \upsilon_d]{} & f^a s & \xrightarrow[z_g]{} & gt
\end{array}
$$

commutes for all $d \in \operatorname{ob} D$. Indeed, we can fill this to create the following commutative diagram:

$$
\begin{array}{ccccccccccc}
fd & = & gid & \xrightarrow{\tilde{a}} & g^a yid & & & & & & \\
\Big\downarrow{\scriptstyle \tilde{a}} & & \Big\downarrow{\scriptstyle \tilde{a}} & & \Big\downarrow{\scriptstyle \tilde{a}} & & & & {\scriptstyle (g^a yid \,=\, g^a yid)} & & \\
f^a yd & = & (gi)^a yd & \xrightarrow{\tilde{a}} & (g^a yi)^a yd & \xrightarrow[\hat{a}]{} & g^a (yi)^* yd & \xrightarrow[\eta^{-1}]{} & g^a yid & \xrightarrow{\tilde{a}^{-1}} & gid \\
\Big\downarrow{\scriptstyle \upsilon_d} & & \Big\downarrow{\scriptstyle \upsilon_d} & & \Big\downarrow{\scriptstyle \upsilon_d} & & \Big\downarrow{\scriptstyle \upsilon_d} & & \Big\downarrow{\scriptstyle g^a y!} & & \Big\downarrow{\scriptstyle g!} \\
f^a s & = & (gi)^a s & \xrightarrow[\tilde{a}]{} & (g^a yi)^a s & \xrightarrow[\hat{a}]{} & g^a (yi)^* s & \xrightarrow[g^a !]{} & g^a yt & \xrightarrow[\tilde{a}^{-1}]{} & gt
\end{array}
$$

and so $z_g : f^a s \to gt$ is indeed a map of cocones.

(4) Cocone maps from c to g

Define a functor $h : \mathbb{D}^t \to PD$ by

$$hi := y, \;\; h(d \xrightarrow{!} t) := yd \xrightarrow{\upsilon_d} s.$$

Take a cocone g and consider natural transformations $\beta : f^a h \to g$ for which

$$(\beta \cdot i : f^a hi \to gi) = (\tilde{a}_f^{-1} : f^a y \to f).$$

These are determined by their component β_t, which must satisfy the naturality condition

$$\begin{array}{ccccc} f^a yd & \xrightarrow{\tilde{a}^{-1}} & fd & = & gid \\ {\scriptstyle f^a \upsilon_d}\downarrow & & & & \downarrow{\scriptstyle g!} \\ f^a s & & \xrightarrow[\beta_t]{} & & gt \end{array}$$

which states precisely that β_t is a map of cocones from c to g. Hence we have a correspondence between

- natural transformations $\beta : f^a h \to g$ such that $\beta \cdot i = \alpha^{-1}$, and
- maps of cocones from c to g.

(5) Uniqueness

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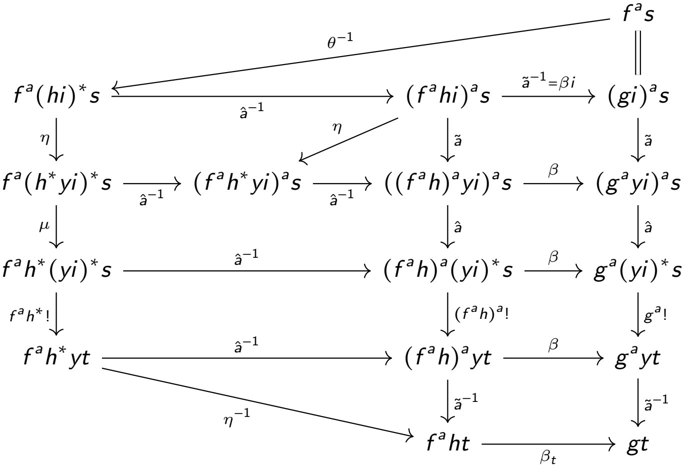
Let $\beta : f^a h \to g$ be a map of cocones from c to g. We construct the following commutative diagram:



(5, cont.) Uniqueness

The diagram demonstrates that the clockwise composite z_g is equal to the anticlockwise composite

$$f^a s \xrightarrow{f^a\theta^{-1}} f^a y^* s = f^a(hi)^* s \xrightarrow{f^a\eta} f^a(h^* yi)^* s \xrightarrow{f^a\mu} f^a h^*(yi)^* s$$
$$\xrightarrow{f^a h^* !} f^a h^* yt \xrightarrow{f^a\eta^{-1}} f^a ht \xrightarrow{\beta_t} gt.$$

But by functoriality this composite is of the form

$$f^a s \xrightarrow{f^a(\dots)} f^a s \xrightarrow{\beta_t} gt$$

for some map $s \to s$. But since s is terminal in PD, the only such map is 1_s. Hence again by functoriality we have

$$z_g = \beta_t f^a(1_s) = \beta_t 1_{f^a s} = \beta_t.$$

So indeed the map of cocones $z_g : f^a s \to gt$ is unique, which implies

$$f^a s \cong \operatorname{colim} f.$$

Hence every presheaf pseudoalgebra $(A, {}^a; \tilde{a}, \hat{a})$ is cocomplete.

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Pseudomorphism $\implies$ Cocontinuous

Corollary

Let $(f,\bar{f})\colon(A,{}^{a};\tilde{a},\hat{a})\to(B,{}^{b};\tilde{b},\hat{b})$ *be a pseudomorphism of* P*-pseudoalgebras. Then* f *preserves all small colimits, in that*

$$f\operatorname{colim} g \cong \operatorname{colim} fg$$

for $D\in\mathrm{CAT}$ *and* $g : D\to A$*. In particular,* f^{a} *preserves all colimits for any* $f : D\to A$*.*

Proof.

We have the following chain of natural isomorphisms:

$$f\operatorname{colim} g \xrightarrow{\sim} fg^{a}\operatorname{colim} y \xrightarrow{\bar{f}_g^{-1}} (fg)^{b}\operatorname{colim} y \xrightarrow{\sim} \operatorname{colim} fg,$$

where the isomorphisms marked $\sim$ exist by the previous theorem. □

P is Algebraically Lax-idempotent After All

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Corollary

The presheaf construction is algebraically lax-idempotent.

Proof.

Let $(A, {}^a; \tilde{a}, \hat{a})$ be a P-pseudoalgebra. Let $D \in \mathrm{Cat}$ and $f : D \to A$. Then $f^a : PD \to A$ preserves all small colimits by the previous corollary and $f^a y$ is isomorphic to f. Since PD is the free cocompletion of D, f^a is therefore the left Kan extension of f along y (Kelly, 1982).
So P is algebraically lax-idempotent. □

Coda

The last result gives us hope that there might be a more conceptual proof that Alg(P) is COC. For example, there might be abstract properties of the inclusion $J\colon \mathrm{Cat} \to \mathrm{CAT}$ that force Alg(P) to be biequivalent to Alg(P_s)—in one dimension, it is enough for the relative monad to be along a dense functor (Arkor 2022).