Pseudocommutativity and Lax Idempotency for Relative Pseudomonads

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1 Relative Pseudomonads

Definition 1.1. (Relative pseudomonad) Let $\mathbb{C}, \mathbb{D}$ be 2-categories and let $J : \mathbb{D} \to \mathbb{C}$ be a 2-functor. A *relative pseudomonad* $(T, i, ^*; \eta, \mu, \theta)$ along J comprises

- for $X \in \operatorname{ob} \mathbb{D}$ an object $TX \in \operatorname{ob} \mathbb{C}$ and map $i_X : JX \to TX$ (called a *unit map*), and
- for $X, Y \in \operatorname{ob} \mathbb{D}$ a functor

$$\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$$

(called an *extension functor*).

The units and extensions furthermore come equipped with three families of 2-cells

- $\eta_f : f \to f^* i_X$ for $f : JX \to TY$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $g : JX \to TY$, $f : JY \to TZ$, and
- $\theta_X : (i_X)^* \to 1_{TX}$ for $X \in \operatorname{ob} \mathbb{D}$,

satisfying two coherence conditions.

Definition 1.2. (2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \operatorname{ob} \mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, ..., X_n; Y)$ for all $n \geq 0$ and objects $X_1, ..., X_n, Y$ which we call a *hom-category*; an object of the hom-category $\mathbb{C}(X_1, ..., X_n; Y)$ is denoted by $f : X_1, ..., X_n \to Y$,
3. an identity multimorphism functor $\mathbf{1}_X : \mathbb{1} \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \operatorname{ob} \mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, ..., X_n; Y) \times \mathbb{C}(W_{1,1}, ..., W_{1,m_1}; X_1) \times ... \times \mathbb{C}(W_{n,1}, ..., W_{n,m_n}; X_n)$$

$$\to \mathbb{C}(W_{1,1}, ..., W_{n,m_n}; Y)$$
$$(f, g_1, ..., g_n) \mapsto f \circ (g_1, ..., g_n)$$

for all arities $n, m_1, ..., m_n$ and objects $Y, X_1, ..., X_n, W_{1,1}, ..., W_{n,m_n}$ in $\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

2 Pseudocommutativity

Definition 2.1. (Strong relative pseudomonad) Let $\mathbb{C}$ and $\mathbb{D}$ be 2-multicategories and let $J : \mathbb{D} \to \mathbb{C}$ be a (unary) 2-functor between them. A *strong relative pseudomonad* $(T, i, ^t; \tilde{t}, \hat{t}, \theta)$ along J comprises:

- for every object X in $\mathbb{D}$ an object TX in $\mathbb{C}$ and unit map $i_X : JX \to TX$,
- for every n, index $1 \leq i \leq n$, objects $B_1, ..., B_{i-1}, B_{i+1}, ..., B_n$ in $\mathbb{C}$ and objects X, Y in $\mathbb{D}$ a functor
$$\mathbb{C}(B_1, ..., B_{i-1}, JX, B_{i+1}, ..., B_n; TY) \xrightarrow{(-)^{t_i}} \mathbb{C}(B_1, ..., B_{i-1}, TX, B_{i+1}, ..., B_n; TY)$$
called the *strength* (in the ith argument) and which is pseudonatural in all arguments, along with three natural families of invertible 2-cells:
- $\tilde{t}_f : f \to f^{t_j} \circ_j i$,
- $\hat{t}_{f,g} : (f^{t_j} \circ_j g)^{t_{j+k-1}} \to f^{t_j} \circ_j g^{t_k}$, and
- $\theta_X : (i_X)^{t_1} \to 1_{TX}$
for $f : B_1, ..., JX, ..., B_n \to TY$ and $g : C_1, ..., JW, ..., C_m \to TX$, satisfying two coherence conditions.

Proposition 2.2. *Let T be a strong relative pseudomonad along multicategorical 2-functor $J : \mathbb{D} \to \mathbb{C}$. Then T is a pseudo-multifunctor $T : \mathbb{D} \to \mathbb{C}$, defining the action of T on 1-cells by the functors*

$$\mathbb{D}(X_1, ..., X_n; Y) \xrightarrow{(i_Y \circ J-)^{t_1 t_2 ... t_n}} \mathbb{C}(TX_1, ..., TX_n; TY),$$

so that for $f : X_1, ..., X_n \to Y$ we have

$$Tf := (i_Y \circ Jf)^{t_1 t_2 ... t_n} = \bar{f}^{t_1, ..., t_n} : TX_1, ..., TX_n \to TY.$$

Definition 2.3. (Pseudocommutative monad) Let T be a strong relative pseudomonad. We say that T is *pseudocommutative* if for every pair of indices $1 \leq j < k \leq n$ and map

$$f : B_1, ..., B_{j-1}, JX, B_{j+1} ..., B_{k-1}, JY, B_{k+1}, ..., B_n \to TZ$$

we have an invertible 2-cell

$$\gamma_f : f^{t_k t_j} \to f^{t_j t_k} : B_1, ..., TX, ..., TY, ..., B_n \to TZ$$

which is pseudonatural in all arguments and which satisfies five coherence conditions (two for $\tilde{t}$, two for $\hat{t}$, and a braiding condition).

Definition 2.4. (Multicategorical relative pseudomonad) Let $\mathbb{C}, \mathbb{D}$ be 2-multicategories and let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. We say T is a *multicategorical relative pseudomonad* if

- T is a pseudo-multifunctor, and
- The unit and extension of T are compatible with the multicategorical structure.

Theorem 2.5. *Let T be a strong relative pseudomonad along multicategorical 2-functor $J : \mathbb{D} \to \mathbb{C}$. Suppose T is pseudocommutative. Then T is a multicategorical relative pseudomonad.*

3 Lax Idempotency

Definition 3.1. (Lax-idempotent strong relative pseudomonad) Let $J : \mathbb{D} \to \mathbb{C}$ be a pseudo-multifunctor and let T be a strong relative pseudomonad along J. We say T is a *lax-idempotent strong relative pseudomonad* if the strength is left adjoint to precomposition with the unit. That is, we have an adjunction

$$\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \underset{- \circ_j i_X}{\overset{(-)^{t_j}}{\rightleftarrows}} \mathbb{C}(B_1, ..., TX, ..., B_n; TY) \quad (\perp)$$

for every $1 \leq j \leq n$ and objects $B_1, ..., B_{j-1}, JX, B_{j+1}, ..., B_n; TY$ whose unit $- \implies (-)^{t_j} \circ_j i$ has components

$$\tilde{t}_f : f \to f^{t_j} \circ_j i_X$$

obtained from the strong structure (again the unit is invertible).

Theorem 3.2. *Let $T : \mathbb{D} \to \mathbb{C}$ be a lax-idempotent strong relative pseudomonad. Then T is pseudocommutative, with a pseudocommutativity whose components $\gamma_g : g^{ts} \to g^{st}$ are given by the composite*

$$g^{ts} \xrightarrow{(\tilde{s}_g)^{ts}} (g^s \circ_s i)^{ts} \xrightarrow{\sim} (g^{st} \circ_s i)^s \xrightarrow{\sigma_{g^{st}}} g^{st}.$$

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