UNIVERSITY OF LEEDS

Property-like Structures and Relative Pseudomonads

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Outline

1. Idempotent monads and 'structure vs. properties'
2. Relative pseudomonads
3. Equivalent definitions of
   lax-idempotent relative pseudomonads

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Idempotent Monads

Definition

A monad (T, i, m) on a category $\mathbb{C}$ is called idempotent if the multiplication natural transformation

$$m : TT \to T$$

is an isomorphism (equivalently, if $m_A : TTA \to TA$ is an isomorphism for all $A \in \mathrm{ob}\,\mathbb{C}$).

Idempotent Monads

Example

The abelianisation functor $T : \mathrm{Grp} \to \mathrm{Grp}$ defined by

$$TG := G/[G, G]$$

is an idempotent monad, because its multiplication

$$m_G : TTG \to TG$$

is given by the isomorphism

$$TG/[TG, TG] \xrightarrow{\sim} TG;$$

since TG is abelian the commutator subgroup $[TG, TG]$ is trivial.

Equivalent Conditions

Proposition

(e.g. Borceux 1994) The following conditions on a monad $T : \mathbb{C} \to \mathbb{C}$ are equivalent:

1. *The multiplication $m : TT \to T$ is an isomorphism;*
2. *For every T-algebra (A, a) the T-action $a : TA \to A$ is an isomorphism;*
3. *The forgetful functor $U : T\text{-}Alg \to \mathbb{C}$ is fully faithful.*

Part (2) of the proposition implies that any object of $\mathbb{C}$ has at most one algebra structure: an object A has an algebra structure if and only if the unit $i_A : A \to TA$ is invertible, in which case $a = (i_A)^{-1} : TA \to A$ is the unique structure on it.

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Structure vs. Properties

- 'An abelian group is a group with the property of being abelian.'
- 'A group is a set equipped with the structure of being a group.'

Can we formalise the difference between objects enjoying 'extra properties' and objects carrying 'extra structure'? Category theory gives us such a formalisation (due to Baez-Bartels-Dolan 1998) in terms of forgetful functors.

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Forgetful Functors

Definition

- A functor $U : \mathbb{D} \to \mathbb{C}$ forgets only properties if U is fully faithful;
- A functor $U : \mathbb{D} \to \mathbb{C}$ forgets at most structure if U is faithful.

For example, being abelian is a property of groups, and accordingly the forgetful functor $U : \mathrm{AbGrp} \to \mathrm{Grp}$ is fully faithful, while being a group is a structure on a set, and accordingly the forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ is faithful but not full.

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Idempotent Monads Encode Properties

Using this framework, we can see that (by part (3) of the previous proposition) if T is an idempotent monad then the forgetful functor

$$U : T\text{-Alg} \to \mathbb{C}$$

is fully faithful and hence forgets only properties. That is to say, a T-algebra is just an object of $\mathbb{C}$ having some extra properties.

An equivalent way to say this is that T-Alg is a reflective subcategory of $\mathbb{C}$, whose embedding-reflection adjunction $F \dashv U$ gives the original idempotent monad.

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Recap

Let (T, i, m) be an idempotent monad on $\mathbb{C}$. Then we have that

- the forgetful functor $U : T\text{-Alg} \to \mathbb{C}$ is fully faithful, and
- a T-algebra structure a for an object A of $\mathbb{C}$ is necessarily the inverse of the unit $i_A : A \to TA$; and only exists if this inverse does.

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Now in 2D!

In moving to the two-dimensional setting, we now consider

- pseudomonads $(T, i, m; \eta, \mu, \theta)$ on a bicategory $\mathbb{C}$, and
- pseudo-algebras $(A, a; \tilde{a}, \hat{a})$ for T.
- algebra morphisms $(f, \bar{f}) : (A, a) \to (B, b)$ which can be either lax, pseudo- ($\bar{f}$ is invertible) or strict ($\bar{f}$ is the identity), giving three bicategories written

$$\text{Ps-}T\text{-Alg}_l, \ \text{Ps-}T\text{-Alg}, \ \text{Ps-}T\text{-Alg}_s.$$

Moreover, in this setting we encounter a natural 'intermediate' situation between having a property and having a structure.

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Property-like Structures (Kelly-Lack 1997)

Consider the 2-category of small categories Cat; compare the following constructions on Cat:

- the pseudomonad whose algebras are the monoidal categories ($T\mathbb{C}$ having objects lists of objects of $\mathbb{C}$, etc); and
- the pseudomonad whose algebras are finitely-cocomplete categories—the finite cocompletion monad ($T\mathbb{C}$ is the full subcategory of Psh $\mathbb{C}$ consisting of finite colimits of representables).

A given category may be equipped with many monoidal structures (e.g. Set either with products or coproducts), but colimits are essentially unique (that is, unique up to unique isomorphism). Thus we say that having all finite colimits is an example of a property-like structure.

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Equivalent Conditions

We want a characterisation of pseudomonads encoding property-like structure that mimics our characterisation of idempotent monads encoding properties.

Definition

(Kelly-Lack 1997) A pseudomonad T on a bicategory is called lax-idempotent if it satisfies either of the following equivalent definitions:

1. the forgetful (1)-functor $U : \text{Ps-}T\text{-Alg}_l \to \mathbb{C}$ is fully faithful,
2. a T-pseudoalgebra structure for an object A of $\mathbb{C}$ is necessarily part of an adjunction $a \dashv i_A : A \to TA$ whose counit $ai_A \to 1_A$ is invertible (given by the pseudoalgebra structure $(\tilde{a})^{-1} : ai_A \to 1_A$).

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Notes

We can phrase (1) more explicitly as

> *Given two pseudoalgebras (A, a) and (B, b), and a map $f : A \to B$, there is a unique way to give f the structure of a lax algebra morphism $(f, \bar{f}) : (A, a) \to (B, b)$.*

Also see that in (2) isomorphism has been replaced with adjunction, meaning algebra structure is now only unique up to unique isomorphism, as we expect.

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Relative Monads

We can generalise monads (T, i, m) for T an endofunctor on $\mathbb{C}$ to <u>relative monads</u> $(T, i, ^*)$ for T a functor $T : \mathbb{D} \to \mathbb{C}$ along a given functor $J : \mathbb{D} \to \mathbb{C}$ (Altenkirch-Chapman-Uustalu 2015).

Example

Fix a field $\mathbb{F}$. Let J : FinSet → Set be the inclusion and define

$$T : \text{FinSet} \to \text{Set} : X \mapsto \{X \xrightarrow{f} \mathbb{F}\}.$$

Then suitable choices of i and $(-)^*$ give T the structure of a relative monad, whose algebras are the vector spaces.

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Relative Pseudomonads (Fiore-Gambino-Hyland-Winskel)

Let $\mathbb{C}, \mathbb{D}$ be bicategories (though we will suppress the bicategory isomorphisms for this talk) and let $J : \mathbb{D} \to \mathbb{C}$ be a pseudofunctor.

Definition

A relative pseudomonad $(T, i, {}^*; \mu, \eta, \theta)$ along $J : \mathbb{D} \to \mathbb{C}$ consists of:

- for every object $A \in \mathbb{D}$ an object $TA \in \mathbb{C}$ and morphism $i_A : JA \to TA$ (the unit), and
- a natural family of functors $(-)^* : \mathbb{C}(JA, TB) \to \mathbb{C}(TA, TB)$ for every pair of objects $A, B \in \mathbb{D}$,

along with three natural families of invertible 2-cells:

- associativity $\mu_{g,f} : (g^* f)^* \to g^* f^*$ for $f : JA \to TB$ and $g : JB \to TC$,
- right unit $\eta_f : f \to f^* i_A$ for $f : JA \to TB$, and
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satisfying two coherence diagrams of 2-cells.

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- for every object $A \in \mathbb{D}$ an object $TA \in \mathbb{C}$ and morphism $i_A : JA \to TA$ (the unit), and
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along with three natural families of invertible 2-cells:

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satisfying two coherence diagrams of 2-cells.

Relative Pseudomonads (Fiore-Gambino-Hyland-Winskel)

Let $\mathbb{C}, \mathbb{D}$ be bicategories (though we will suppress the bicategory isomorphisms for this talk) and let $J : \mathbb{D} \to \mathbb{C}$ be a pseudofunctor.

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The presheaf relative pseudomonad

Example

Let $J : \mathsf{Cat} \to \mathsf{CAT}$ be the inclusion of small categories into locally small categories. Then the presheaf construction

$$\mathsf{Psh}\,\mathbb{C} := [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$$

can be given the structure of a relative pseudomonad, with unit maps

$$i_{\mathbb{C}} := Y_{\mathbb{C}} : \mathbb{C} \to \mathsf{Psh}\,\mathbb{C}$$

and for $F : \mathbb{C} \to \mathsf{Psh}\,\mathbb{D}$ the extension

$$F^* := \mathsf{Lan}_Y F : \mathsf{Psh}\,\mathbb{C} \to \mathsf{Psh}\,\mathbb{D}.$$

This example generalises the finite cocompletion pseudomonad (and the cocompletion pseudomonads for various other classes of colimit).

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Algebras over a Relative Pseudomonad

Definition

Let T be a relative pseudomonad along pseudofunctor $J : \mathbb{D} \to \mathbb{C}$. Then a pseudoalgebra $(A, {}^a; \tilde{a}, \hat{a})$ over T consists of:

- an object $A \in \mathbb{C}$, and
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along with two natural families of 2-cells:

- $\tilde{a}_f : f \to f^a i_X$ for $f : JX \to A$, and
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Free Pseudoalgebras

Example

For any object $Y \in \mathrm{ob}\,\mathbb{D}$, the object $TY \in \mathrm{ob}\,\mathbb{C}$ has the structure of a pseudoalgebra. Its algebra extension is given by the monad extension

$$(-)^* : \mathbb{C}[JX, TY] \to \mathbb{C}[TX, TY]$$

and its structural 2-cells are given by

$$\eta_f : f \to f^* i_X,\ \ \mu_{g,f} : (g^* f)^* \to g^* f^*.$$

Morphisms of Pseudoalgebras

Let $(A, {}^{a})$ and $(B, {}^{b})$ be pseudoalgebras over a relative pseudomonad T. A lax morphism

$$(f, \bar{f}) : (A, {}^{a}) \to (B, {}^{b})$$

consists of

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If the components of $\bar{f}$ are invertible we say $(f, \bar{f})$ is a pseudomorphism; if they are identities we say it is a strict morphism.

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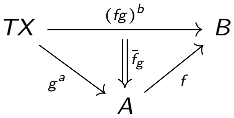

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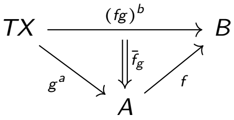

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Morphisms of Pseudoalgebras, cont.

Example

Let $(A, {}^a)$ be a pseudoalgebra and $f : JY \to A$ a map with extension $f^a : TY \to A$. Since TY and A are both pseudoalgebras, it makes sense to ask whether f^a has a lax morphism structure. Such a structure would have components

$$\overline{f^a}_g : (f^a g)^a \to f^a g^*$$

for $g : JX \to TY$. But the pseudoalgebra structure on A gives us such a map:

$$\overline{f^a}_g := \hat{a}_{f,g} : (f^a g)^a \to f^a g^*$$

and one can check that this does give a lax morphism structure—in fact a pseudomorphism structure, since $\hat{a}_{f,g}$ is invertible for all g.

Recap

Our aim is to prove the equivalence of the two conditions

1. Every map between pseudoalgebras has a unique lax morphism structure;
2. Every pseudoalgebra structure is left adjoint to the monad unit

in the relative setting.

We now have the tools necessary to say what we mean by this (to wit, we have defined pseudoalgebras over a relative pseudomonad and lax morphisms between them).

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Lax-idempotent Relative Pseudomonads

Let's translate the two conditions into the relative setting. The first requires almost no change:

Definition
A relative pseudomonad T is property-like if the forgetful (1)-functor $U : \text{Ps-}T\text{-Alg}_l \to \mathbb{C}$ is fully faithful. That is, if every map $f : A \to B$ between pseudoalgebras $(A, {}^a)$, $(B, {}^b)$ has a unique lax morphism structure $(f, \bar{f}) : (A, {}^a) \to (B, {}^b)$.

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Lax-idempotent Relative Pseudomonads, cont.

The second condition changes somewhat more in the relative setting, since the algebra structure is defined differently.

Definition

A relative pseudomonad T has algebra adjoint to unit if, for every pseudoalgebra $(A, {}^a)$, the algebra structure map $(-)^a : \mathbb{C}(JX, A) \to \mathbb{C}(TX, A)$ for $X \in \mathbb{D}$ is part of an adjunction

$$(\tilde{a}, \alpha) : (-)^a \dashv (-) \circ i_X$$

with invertible unit $1 \implies (-)^a i_X$ whose components are given by the pseudoalgebra structure on A:

$$\tilde{a}_f : f \to f^a i_X .$$

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The Newest Result

Theorem

Let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. Then the following are equivalent:

(i) *T is property-like,*

(ii) *T has algebra adjoint to unit.*

If T satisfies either condition (and thus both), we say T is lax-idempotent.

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The Proof Begins

We first show that $(i) \implies (ii)$. So assume that T is property-like, and let $(A, {}^a)$ be a pseudoalgebra. We need to show that we have an adjunction

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for which it suffices to define a counit

$$\alpha : (-i_X)^a \implies 1 : \mathbb{C}[TX, A] \to \mathbb{C}[TX, A]$$

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The components of this hypothetical counit would be

$$\alpha_h : (hi_X)^a \to h$$

for $h : TX \to A$. Now we use that T is property-like; TX and A are both pseudoalgebras and so h has a unique lax morphism structure $\bar{h}$ with components

$$\bar{h}_g : (hg)^a \to hg^*.$$

Using this we can define our counit by the composite

$$(hi_X)^a \xrightarrow{\bar{h}_{i_X}} hi_X^* \xrightarrow{h\theta_X} h$$

(and one can check that these components form a natural transformation $(-i_X)^a \implies 1$).

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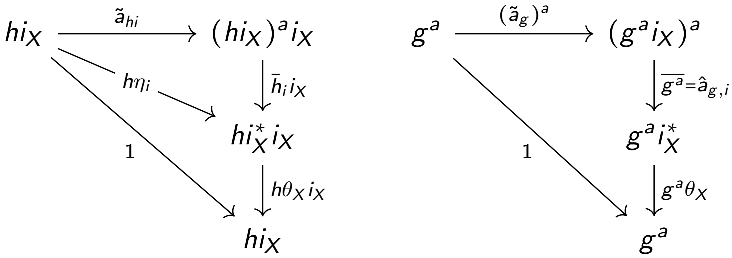
The triangle identities have components



and the left triangle decomposes as a coherence diagram for $\bar{h}$ being a lax morphism structure and a coherence diagram for the pseudomonad, while the right triangle is exactly one of the coherence diagrams making $(A, {}^a)$ a pseudoalgebra.

Proof, cont.

Thus we do indeed have an adjunction

$$(\tilde{a}, \alpha) : (-)^a \dashv (-) \circ i_X.$$

We now go on to show that $(ii) \implies (i)$. Let $(A, ^a)$ and $(B, ^b)$ be pseudoalgebras and let $f : A \to B$ be a map in $\mathbb{C}$. We need to show that there is a lax morphism structure on f and that it is unique.

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Uniqueness

We first show uniqueness. To that end, suppose $\bar{f}$ is a lax morphism structure for f; it has components

$$\bar{f}_g : (fg)^b \to fg^a$$

for $g : JX \to A$. Using part (ii) [specifically, using $(-)^b \dashv (-) \circ i_X$] we know that these have transposes (which we will denote by ϕ_g)

$$\phi_g : fg \to fg^a i.$$

We can write ϕ_g and $\bar{f}_g$ in terms of each other as follows:

$$fg \xrightarrow{\phi_g} fg^a i \qquad = \qquad fg \xrightarrow{\tilde{b}_{fg}} (fg)^b i \xrightarrow{\bar{f}_g i} fg^a i$$

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One of the coherence diagrams for $\bar{f}$ being a lax morphism structure is

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Existence

It remains only to show that

$$(fg)^b \xrightarrow{(f\tilde{a}_g)^b} (fg^a i)^b \xrightarrow{\beta_{fg^a}} fg^a$$

is indeed a lax morphism structure. There are two coherence conditions to check; the first diagram

$$\begin{array}{ccc} fg & \xrightarrow{\tilde{b}_{fg}} & (fg)^b i_X \\ & {\scriptstyle f\tilde{a}_g} \searrow & \downarrow {\scriptstyle \bar{f}_g i_X} \\ & & fg^a i_X \end{array}$$

commutes by construction of $\bar{f}$: one composite is ϕ_g while the other is 'the transpose of $\bar{f}_g$'.

Existence, cont.

The other coherence condition is that the pentagon

$$\begin{array}{ccc} ((fg)^b h)^b & \xrightarrow{\hat{b}_{fg,h}} & (fg)^b h^* \\ {\scriptstyle (\bar{f}_g h)^b}\big\downarrow & & \\ (fg^a h)^b & & \big\downarrow{\scriptstyle \bar{f}_g h^*} \\ {\scriptstyle \bar{f}_{g^a h}}\big\downarrow & & \\ f(g^a h)^a & \xrightarrow[f\hat{a}_{g,h}]{} & fg^a h^* \end{array}$$

commutes.

Using our definition of $\bar{f}$ we construct the diagram

$$
\begin{array}{ccccc}
((fg)^b h)^b & \xrightarrow{((fg)^b \eta_h)^b} & ((fg)^b h^* i)^b & \xrightarrow{\beta_{(fg)^b h^*}} & (fg)^b h^* \\
{\scriptstyle ((f\tilde{a}_g)^b h)^b}\downarrow & & {\scriptstyle ((f\tilde{a}_g)^b h^* i)^b}\downarrow & & \downarrow{\scriptstyle (f\tilde{a}_g)^b h^*} \\
((fg^a i)^b h)^b & \xrightarrow{((fg^a i)^b \eta_h)^b} & ((fg^a i)^b h^* i)^b & \xrightarrow{\beta_{(fg^a i)^b h^*}} & (fg^a i)^b h^* \\
{\scriptstyle (\beta_{fg^a} h)^b}\downarrow & & {\scriptstyle (\beta_{fg^a} h^* i)^b}\downarrow & & \downarrow{\scriptstyle \beta_{fg^a} h^*} \\
(fg^a h)^b & \xrightarrow{(fg^a \eta_h)^b} & (fg^a h^* i)^b & \xrightarrow[\beta_{fg^a h^*}]{} & fg^a h^* \\
 & \searrow{\scriptstyle (f\tilde{a}_{g^a h})^b} & \uparrow{\scriptstyle (f\hat{a}_{g,h} i)^b} & & \uparrow{\scriptstyle f\hat{a}_{g,h}} \\
 & & (f(g^a h)^a i)^b & \xrightarrow[\beta_{f(g^a h)^a}]{} & f(g^a h)^a
\end{array}
$$

which comprises a total of five naturality squares and one coherence triangle from the pseudoalgebra structure on A.

The anticlockwise composite is exactly what we want; for the clockwise composite, we need the composite of the two maps along the top of the diagram:

$$((fg)^b h)^b \xrightarrow{((fg)^b \eta_h)^b} ((fg)^b h^* i_X)^b \xrightarrow{\beta_{(fg)^b h^*}} (fg)^b h^*$$

to be equal to

$$((fg)^b h)^b \xrightarrow{\hat{b}_{fg,h}} (fg)^b h^*.$$

But they are both equal to the transpose of

$$(fg)^b h \xrightarrow{(fg)^b \eta_h} (fg)^b h^* i$$

and are therefore themselves equal.

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Proof (end)

Hence indeed $\bar{f}$ as we have defined it is a lax morphism structure for $f : A \to B$, and thus we have shown the two conditions on T are equivalent. □

Conclusion and Future Work

We gave two characterisations of lax-idempotent relative pseudomonads and showed they were equivalent, extending a result of Kelly-Lack. From here I intend to

- extend a result of Lopez-Franco and show that lax-idempotent relative pseudomonads are pseudocommutative;
- apply the theory of lax-idempotent relative pseudomonads to the case of presheaves, whose bicategory of pseudoalgebras and algebra pseudomorphisms turns out to be the 2-category $\underline{\mathrm{COC}}$ of locally-small co-complete categories.

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Thank You! ☺