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Varieties of relative monad

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Introduction

For the purposes of this presentation, let V be a symmetric monoidal closed category with compatible

- tensor product $- \otimes - : V \times V \to V$ with monoidal structure I, α, λ, ρ,
- internal hom $[-,-] : V^{op} \times V \to V$ with closed structure I, L, i, j, and
- symmetry with components $\sigma_{A,B} : A \otimes B \to B \otimes A$,

so that for all $A \in \mathrm{ob}\, V$ we have an adjunction

$$- \otimes A \dashv [A, -].$$

Enriched and Strong Monads (Kock 1970)

A monad (T, η, μ) is

- enriched if for all A, B we have a map $T_{A,B} : [A, B] \to [TA, TB]$ compatible with the closed structure, and
- strong if for all A, B we have a map $t_{A,B} : A \otimes TB \to T(A \otimes B)$ compatible with the monoidal structure.

Proposition

A monad is enriched if and only if it is strong, with correspondence:

$t_{A,B}$ *is the transpose of* $A \xrightarrow{con} [B, A \otimes B] \xrightarrow{T} [TB, T(A \otimes B)]$,

$T_{A,B}$ *is the transpose of* $[A, B] \otimes TA \xrightarrow{t} T([A, B] \otimes A) \xrightarrow{Tev} TB$.

Commutative Monads (Kock 1970)

We can define a costrength $s_{A,B} : TA \otimes B \to T(A \otimes B)$ by

$$s_{A,B} = T\sigma_{B,A} \circ t_{B,A} \circ \sigma_{TA,B}.$$

Now a strong monad is commutative if the map

$$\phi_{A,B} : TA \otimes TB \to T(A \otimes B)$$

— defined to be the composite

$$TA \otimes TB \xrightarrow{s} T(TA \otimes B) \xrightarrow{Tt} TT(A \otimes B) \xrightarrow{\mu} T(A \otimes B)$$

— is equal to the composite

$$TA \otimes TB \xrightarrow{t} T(A \otimes TB) \xrightarrow{Ts} TT(A \otimes B) \xrightarrow{\mu} T(A \otimes B).$$

Symmetric Monoidal Monads (Kock 1970)

It can be shown that the map $\phi_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined above gives T the structure of a lax monoidal functor. If T is a lax monoidal functor and furthermore η, μ are monoidal natural transformations, we say T is a monoidal monad.
If moreover we have

$$T\sigma_{A,B} \circ \phi_{A,B} = \phi_{B,A} \circ \sigma_{TA,TB},$$

we say T is a symmetric monoidal monad.

Commutative iff Symmetric Monoidal

Proposition

A monad $T : V \to V$ is commutative if and only if it is symmetric monoidal; given a commutative monad we can define structure maps via

$$\phi. := \eta_I : I \to TI, \ \phi_{A,B} := \mu_{A\otimes B} \circ Tt_{A,B} \circ s_{A,TB},$$

and given a symmetric monoidal monad we can define strength and costrength maps by

$$t_{A,B} := \phi_{A,B} \circ (\eta_A \otimes 1_B), \ s_{A,B} := \phi_{A,B} \circ (1_A \otimes \eta_B).$$

Summary of Implications

Kock's work gives

- T enriched $\iff$ T strong,
- T enriched/strong $\implies$ T lax monoidal functor, and
- T commutative $\iff$ T symmetric monoidal.

Relative Monads

Let $\mathbb{C}, \mathbb{D}$ be symmetric monoidal closed categories and let $J : \mathbb{D} \to \mathbb{C}$ be a strict monoidal functor (in applications J is usually even an inclusion).

A relative monad $(T, \eta, (-)^*)$ along J comprises:

- for each object $A \in \operatorname{ob} \mathbb{D}$ an object $TA \in \operatorname{ob} \mathbb{C}$ and morphism $\eta_A : JA \to TA$, and
- an extension $(-)^* : \mathbb{C}(JA, TB) \to \mathbb{C}(TA, TB)$ satisfying
  - $\eta_A^* = 1_{TA}$ for all A,
  - $f^* \circ \eta_A = f$ for all $f : JA \to TB$, and
  - $g^* \circ f^* = (g^* \circ f)^*$ for all $JA \to TB$, $g : JB \to TC$.

It can be shown that, given these constraints, T is a functor $\mathbb{D} \to \mathbb{C}$ and the η_A form a natural transformation $\eta : J \implies T$. Furthermore, a relative monad along 1_C is exactly an ordinary monad.
My work hereon is to define analogous notions of Kock's 'enriched, strong, commutative, symmetric monoidal' for relative monads.

Enriched Relative Monads

A relative monad T along J is <u>enriched</u> if the mapping $(f : JA \to TB) \mapsto (f^* : TA \to TB)$ internalises to a morphism

$$* : [JA, TB] \to [TA, TB],$$

satisfying some coherence diagrams. For example, we require that the diagram

$$\begin{array}{ccc} [JA, TB] & \xrightarrow{\;*\;} & [TA, TB] \\ & \searrow^{1} & \downarrow{\scriptstyle [\eta,1]} \\ & & [JA, TB] \end{array}$$

commutes, corresponding to the equation $f^* \circ \eta = f$.

Strong Relative Monads

A relative monad T along J is strong if it comes equipped with a map

$$t_{A,B} : JA \otimes TB \to T(A \otimes B)$$

satisfying some coherency diagrams. For example, coherency with $(-)^*$ is given by, for all $f : A \to A'$, $g : JB \to TB'$, commutativity of

$$\begin{array}{ccc} JA \otimes TB & \xrightarrow{Jf \otimes g^*} & JA' \otimes TB' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ T(A \otimes B) & \xrightarrow{(t \circ (Jf \otimes g))^*} & T(A' \otimes B') \end{array}$$

where the bottom arrow is the result of applying the extension $(-)^*$ to the composite

$$J(A \otimes B) = JA \otimes JB \xrightarrow{Jf \otimes g} JA' \otimes TB' \xrightarrow{t} T(A \otimes B')$$

Enriched Implies Strong

A relative monad is strong if it is enriched, with strength $t_{A,B} : JA \otimes TB \to T(A \otimes B)$ defined as the transpose of the composite

$$JA \xrightarrow{con} [JB, JA \otimes JB] = [JB, J(A \otimes B)]$$
$$\xrightarrow{[1,\eta]} [JB, T(A \otimes B)] \xrightarrow{*} [TB, T(A \otimes B)].$$

However, things go wrong in the other direction; if we attempt to define the transpose of $* : [JA, TB] \to [TA, TB]$ via $t_{A,B}$, we look for a map

$$[JA, TB] \otimes TA \to TB.$$

Now $[JA, TB]$ is not necessarily of the form JX for some $X \in \mathrm{ob}\,\mathbb{D}$, and so we cannot apply any $t_{X,A}$ to the domain $[JA, TB] \otimes TA$.

T is still lax monoidal

Let T be enriched (and therefore strong). We have costrength $s_{A,B}$ as before and we can now define a map

$$\phi_{A,B} : TA \otimes TB \to T(A \otimes B)$$

in the relative setting, as the transpose of the composite

$$TA \xrightarrow{con} [JB, TA \otimes JB] \xrightarrow{[1,s]} [JB, T(A \otimes B)] \xrightarrow{*} [TB, TA \otimes TB].$$

It can be shown that this $\phi_{A,B}$ along with $\phi. := \eta_I : JI = I \to TI$, gives T the structure of a lax monoidal functor $\mathbb{D} \to \mathbb{C}$.

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Commutative Relative Monads

An enriched relative monad T along $J : \mathbb{D} \to \mathbb{C}$ is commutative if we have

$$T\sigma_{A,B} \circ \phi_{A,B} = \phi_{B,A} \circ \sigma_{A,B},$$

where $\phi_{A,B} : TA \otimes TB \to T(A \otimes B)$ is defined (as before) as the transpose of

$$TA \xrightarrow{con} [JB, TA \otimes JB] \xrightarrow{[1,s]} [JB, T(A \otimes B)] \xrightarrow{*} [TB, T(A, \otimes B)].$$

Symmetric Monoidal Relative Monads

We say a relative monad T is monoidal if:

1. As a functor, T is lax monoidal with structure maps $\phi_\cdot$, $\phi_{A,B}$,
2. the maps η_A satisfy
   a. $\phi_\cdot = \eta_I : JI = I \to TI$,
   b. $\phi_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A\otimes B} : JA \otimes JB = J(A \otimes B) \to T(A \otimes B)$.
3. the extension $(-)^*$ satisfies
   a. $(\phi_\cdot)^* = 1_{TI}$,
   b. $(\phi_{A',B'} \circ (f \otimes g))^* \circ \phi_{A,B} = \phi_{A',B'} \circ (f^* \otimes g^*)$ for all $f : JA \to TA'$ and $g : JB \to TB'$.

   Note that in fact condition (2a) implies (3a).

We say that T is symmetric monoidal if we furthermore have

$$T\sigma_{A,B} \circ \phi_{A,B} = \phi_{B,A} \circ \sigma_{TA,TB}.$$

Commutative Implies Symmetric Monoidal

Theorem

If T is a commutative relative monad, T is a symmetric monoidal relative monad, with structure maps

$$\phi. := \eta_I,\ \phi_{A,B} \text{ the transpose of } * \circ [1, s_{A,B}] \circ con_{TA,JB}.$$

- The symmetry condition follows immediately from the definition of commutativity. Conditions (2a,3a) follow from the above definition of the structure map $\phi.$ and (2b,3b)—after some calculation—from the definition of $\phi_{A,B}$ and commutativity.
- Again we have difficulty going the other way; we cannot define an enrichment $* : [JA, TB] \to [TA, TB]$ merely given that T is symmetric monoidal.

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Summary of Implications for Relative Monads

My work here gives

- T enriched $\implies$ T strong,
- T enriched $\implies$ T lax monoidal functor, and
- T commutative $\implies$ T symmetric monoidal.