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Well-quasiorders and Kruskal's Tree Theorem

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Prerequisites

Some knowledge of order theory is assumed, including the basics of partial orders, total orders and well-orders, along with the theory of countable ordinals up to ε_0. I also assume some knowledge of combinatorics, including the statement of Ramsey's theorem for k-partitions of $\mathbb{N}^{(2)}$.

Initial Definitions and Notation

Definition 1
A quasiordering (or a preordering) $\leq$ on a set X is a reflexive and transitive relation on X. We call a set X equipped with such a relation a quasiorder (or a preorder).

We will write $a < b$ (and say 'a is strictly less than b') if $a \leq b$ and $b \nleq a$. We will also write $a \not\lesseqgtr b$ (and say 'a and b are incomparable') if neither $a \leq b$ nor $b \leq a$.

Examples

Any partial order, total order or well-order is a quasiorder.
Various contexts with a notion of 'embedding' form quasiorders.
For example, we might take:

- (some set of) groups, with $G \leq H$ exactly when there is an injective group homomorphism $G \to H$,
- (some set of) topological spaces, with injective continuous maps,
- (some set of) infinite graphs, with the subgraph relation, or the graph minor relation.

Well-foundedness

Definition 2

- Given a set X, a quasiorder $\leq$ on X is well-founded if every subset $A \subseteq X$ has a minimal element with respect to $\leq$. That is, for each $A \subseteq X$ there exists an $a \in A$ such that for every $b \in A$, $b \nless a$.
- Equivalently (given the Axiom of Dependent Choice, which I will assume), the relation is well-founded if it contains no countably-infinite descending chain $x_0 > x_1 > x_2 > \ldots$ in X.

However, well-foundedness of given quasiorders need not be preserved under lifting operations. For example, $(\mathbb{N}, |)$ is a well-founded quasiorder, but the sequence

$$P_2 > P_3 > P_5 > \ldots$$

where $P_n := \{p \geq n : p \text{ prime}\}$ is an infinite descending sequence in $P(\mathbb{N})$.
So, when is the powerset of a quasiorder well-founded?

Goodness

Take a quasiorder X and consider sequences $\bar{a} : \mathbb{N} \to X$.

- A pair (a_i, a_j) is called good if $i < j$ in $\mathbb{N}$ and $a_i \leq a_j$ in X.
- The whole sequence is called good if it contains a good pair. Otherwise it is bad.

This allows us to define a stronger (as we shall see) notion than well-foundedness for our quasiorders.

Definition 3
A well-quasiorder X is a quasiorder for which every sequence $\bar{a} : \mathbb{N} \to X$ is good. (Henceforth we write 'wqo' for 'well-quasiorder'.)

Examples

- The natural numbers $(\mathbb{N}, \leq)$ with the usual order are a wqo — every well-order is wqo. The integers $(\mathbb{Z}, \leq)$ are not wqo, as the sequence of negative integers

$$0, -1, -2, -3, \ldots$$

is bad, and the naturals $(\mathbb{N}, |)$ under divisibility are not wqo, as the sequence of primes

$$2, 3, 5, 7, 11, \ldots$$

is bad. (These are in essence the only types of bad sequence; see Proposition 1).

- if $(X, \leq)$ is a wqo, then the finite product X^k with componentwise ordering is also wqo (See Proposition 4).
- If X is a finite set, the set X^* of finite strings of elements of X ordered by $a \leq b$ if and only if a is a subsequence of b (for example, $X = \{0, 1\}$, $a = 011$, $b = 01001$) is a wqo (this is called Higman's Lemma). This is a special case of Kruskal's Tree Theorem, which states that if Q is a wqo, then so is the set $T(Q)$ of finite trees labelled with elements of Q, under 'homeomorphic embedding'.

Characterising Well-quasiorders

Proposition 1

Let A be a set with quasiorder $\leq$. Then the following are equivalent:

(i) *A is a well-quasiordering.*

(ii) *A contains no infinite strictly-decreasing sequence, nor an infinite sequence of pairwise-incomparable elements.*

(iii) *Every sequence $\bar{a} : \mathbb{N} \to A$ contains a non-decreasing subsequence $\bar{a}_u$.*

We will show $(i) \implies (ii) \implies (iii) \implies (i)$.

- Let $\bar{a} : \mathbb{N} \to A$ be a sequence in A. By (i), $\bar{a}$ is good, so it contains a good pair $a_i \leq a_j$. Then because of this pair, $\bar{a}$ is neither an strictly-decreasing sequence, nor a sequence of pairwise-incomparable elements.
- Given a sequence $\bar{a} : \mathbb{N} \to A$, partition the two-sets $\{i < j\}$ into three parts P_1, P_2, P_3, given respectively by the trichotomous conditions $a_i \leq a_j$, $a_i > a_j$ and $a_i \not\lessgtr a_j$. Then Ramsey's theorem gives us a infinite monochromatic subset of $\mathbb{N}$.

  But by (ii) this subset cannot be monochromatic in P_2, nor in P_3, and so it must be monochromatic in P_1. This is our non-decreasing subsequence $\bar{a}_u$.

- Let $\bar{a} : \mathbb{N} \to A$ be a sequence in A. By (iii), it contains a non-decreasing subsequence $\bar{a}_u$. In particular, $a_{u(0)} \leq a_{u(1)}$, and this is a good pair, so $\bar{a}$ is a good sequence. $\square$

The Powerset Condition

Proposition 2

Let X be a set with quasiorder $\leq$. Then X is a wqo if and only if the lift $P(X)$ with the relation

$$A \leq B \iff \forall a \in A \, \exists b \in B \; : \; a \leq b$$

is well-founded.

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In both directions we prove the contrapositive.

- Suppose X is not wqo, so we have a bad sequence $\bar{a} : \mathbb{N} \to X$. Define
$$A_i := \{a_j : j \geq i\}.$$
Then
$$A_0 > A_1 > A_2 > ...$$
is a strictly-decreasing sequence in $P(X)$ — if $A_i \leq A_j$ for some $i < j$, there is some $k \geq j > i$ such that $a_i \leq a_k$, contradicting the fact that $\bar{a}$ is bad.

- Conversely, suppose $P(X)$ is not well-founded. Then we have a strictly-decreasing chain of subsets

$$A_0 > A_1 > A_2 > \dots;$$

take for each i some $a_i \in A_i$ such that $a_i \not\leq b$ for all $b \in A_{i+1}$.
Then we claim the sequence (a_i) is bad.
Indeed, let $i < j$. Then since $A_j \leq A_{i+1}$ there is some $c \in A_{i+1}$ with $a_j \leq b$. Then since by construction $a_i \not\leq c$, we must have $a_i \not\leq a_j$.
Hence X is wqo. □

The Minimal Bad Sequence

Definition 4
Let X be a well-founded quasiorder which is not a wqo. A bad sequence $\bar{a} : \mathbb{N} \to X$ is a minimal bad sequence (an MBS) if for each $n \in \mathbb{N}$, a_n is minimal from the set

$$\{a \in X : \text{there is a bad sequence whose first } n \text{ terms are } a_0, ..., a_{n-1}, a\}.$$

We would like to use this notion in some sense like a 'minimal counterexample' in induction proofs. That is, we want to say that every sequence which is 'below' an MBS must be a good sequence.

The Minimal Bad Sequence Lemma

Lemma 3

Let X be a well-founded quasiorder which is not wqo, and let $\bar{a} : \mathbb{N} \to X$ be an MBS. Then the subset

$$Y := \{y \in X : y < a_n \text{ for some } n \in \mathbb{N}\}$$

is wqo.

Let $\bar{b} : \mathbb{N} \to X$ be an arbitrary bad sequence in X. Suppose for the sake of contradiction that every element of $\bar{b}$ is in Y; that is, suppose that for all i there is n such that $b_i < a_n$. Take a pair (i, n) with least possible n and consider the sequence

$$a_0, a_1, ..., a_{n-1}, b_i, b_{i+1}, b_{i+2}, ...$$

— it cannot be bad, or else a_n is not minimal among bad continuations of the initial segment $(a_0, a_1, ..., a_{n-1})$. Thus it contains a good pair, and this must be of the form $a_j \leq b_k$, since $\bar{a}$ and $\bar{b}$ are both bad.

But since $b_k \in Y$, there is some l with $b_k < a_l \implies a_j < a_l$, and by minimality of n we have $j < n \leq l$.
So in fact $a_j < a_l$ is a good pair, contradicting badness of $\bar{a}$. Thus $\bar{b}$ was not in Y, and so every sequence in Y is good.
Hence Y is wqo. $\square$

Well-quasiorders from well-quasiorders

Proposition 4

Let A and B be wqo. Then the following are also wqo:

(i) *the product $A \times B$, given the ordering*

$$(a, b) \leq (a', b') \iff a \leq b \wedge a' \leq b'.$$

(ii) *the set $A^{(<\omega)}$ of finite subsets of A, given the ordering*

$$B \leq C \iff \exists f : B \to C \text{ injective and non-decreasing.}$$

We will show (i), and use this result to prove (ii).

(i) Let $(\bar{a}, \bar{b}) : \mathbb{N} \to A \times B$ be a sequence in $A \times B$, with projections $\bar{a} : \mathbb{N} \to A$ and $\bar{b} : \mathbb{N} \to B$.
By Lemma 1, there is a non-decreasing subsequence $\bar{a}_u$ of $\bar{a}$, since A is a wqo. Since B is also a wqo, the corresponding subsequence $\bar{b}_u$ of $\bar{b}$ has a good pair $b_{u(i)} \leq b_{u(j)}$. Then $(a_{u(i)}, b_{u(i)}) \leq (a_{u(j)}, b_{u(j)})$ and so $(\bar{a}, \bar{b})$ is good.
So $A \times B$ is a wqo.

(ii) Note that the relation $\leq$ on $A^{(<\omega)}$ is reflexive (take $f = 1_B : B \to B$) and transitive (since the composition of non-decreasing functions is itself non-decreasing).
Moreover, it is well-founded: take a subset $\mathcal{A} \subseteq A^{(<\omega)}$, and let $n := \min\{|B| : B \in \mathcal{A}\}$. Since $B \leq C \implies |B| \leq |C|$, a minimal element among the finitely-many elements of size n is minimal in $\mathcal{A}$.

Hence either $A^{(<\omega)}$ is a wqo or we can take an MBS $\bar{B} : \mathbb{N} \to A^{(<\omega)}$. As the empty set is the minimum element in $A^{(<\omega)}$, none of the B_i is empty; pick $b_i \in B_i$ for each i, and write $C_i := B_i \setminus \{b_i\}$.

Note that $C_i < B_i$ (the inclusion is injective and non-decreasing).

Then by the MBS Lemma, the set

$$\mathcal{X} := \{C_i \mid i \in \mathbb{N}\} \subseteq A^{(<\omega)}$$

is wqo.

Now, we know by (i) that $A \times \mathcal{X}$ is a wqo, and thus that the sequence $(\bar{b}, \bar{C})$ is good. But a good pair $(b_i, C_i) \leq (b_j, C_j)$ yields a good pair $B_i \leq B_j$ in $\bar{B}$, contradicting the fact that $\bar{B}$ is a bad sequence.

Hence $A^{(<\omega)}$ is a wqo. □

Basic definitions and notation

One structure to which we can lift a quasiorder is the finite (rooted) tree, which here we can consider as a generalisation of the finite list.

Definition 5
A finite (unlabelled) tree is a finite partially-ordered set t, whose elements are called vertices, such that

- t has a minimum vertex $r = \mathrm{root}(t)$, called the root of t, and
- for every $b \in t$, the set of vertices below b, $\{a : a < b\}$ (the under-set of b), is linearly-ordered.

In this way, we might say that trees 'look like lists when looking down'.

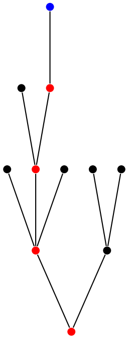

Figure 1: A tree, in which $a \leq b$ if there is a path upwards from a to b. Here the blue vertex has its under-set highlighted in red.

We say 'a is the parent of b' if $a = \max\{x : x < b\}$ (which exists because the set is a finite linear order), and we say that 'b is a child of a' if a is the parent of b (see Figure 2). Note that a vertex can have multiple children.

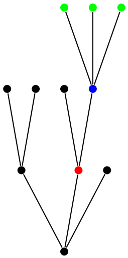

Figure 2: A vertex in blue: its children are in green, and its parent is in red.

For a vertex $b \in t$, the branch at b is the subset $\{a : a \geq b\}$ of t with the induced partial ordering. This is itself a finite tree with root b. In fact, this allows for an inductive definition of trees:

> *A tree is either a single vertex or a finite set of trees with a single vertex below them all.*

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A labelled tree (with labels in the quasiorder Q) is function $\tau : t \to Q$, where t is an unlabelled tree. We say 'a is a vertex of τ with label q' if $a \in t$, $q \in Q$ and $\tau(a) = q$.

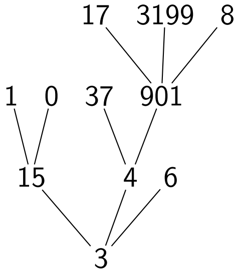

Figure 3: A tree labelled with elements from the quasiorder $Q = \mathbb{N}$.

Maps between trees

Definition 6
A homeomorphic embedding (henceforth a map) $f : t \to u$ between finite trees is an injective function f satisfying, for all $a, b \in t$,

$$f(a \wedge b) = f(a) \wedge f(b),$$

where $a \wedge b$ is the infimum of a and b — that is, the greatest element in both their under-sets. If there is a map $t \to u$ write $t \leq u$; since the composition of maps is again a map, and the identity function is a map, the resulting relation $\leq$ is a quasiorder.

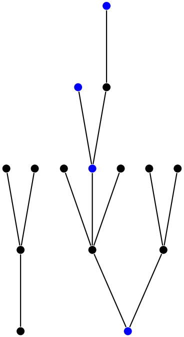

Figure 4: A tree homeomorphically embeds into another; vertices in the range are coloured blue.

Notice that a map f of unlabelled trees is an order-embedding:

$$\begin{aligned} a \leq b &\iff a \wedge b = a \\ &\iff f(a \wedge b) = f(a) \text{ since } f \text{ is injective} \\ &\iff f(a) \wedge f(b) = f(a) \\ &\iff f(a) \leq f(b). \end{aligned}$$

In particular, this means that if f is a surjective map, it is in fact an order-isomorphism.
For labelled trees a non-decreasing homeomorphic embedding (henceforth also called a map) $f : \tau \to \upsilon$ is the corresponding notion: we require that f be a map, considered a a function $t \to u$ (ignoring labels), and that for every vertex a of τ, $\tau(a) \leq \upsilon(f(a))$.

Kruskal's Tree Theorem

We now have all the tools we need to prove the main theorem of this essay.

Theorem 5
The set of finite trees labelled by elements of a well-quasiorder Q, $T(Q)$, is itself a well-quasiorder under homeomorphic embedding.

$T(Q)$ is a well-founded quasiorder

The identity function is a map, and the composition of two maps is again a map: suppose $f : \tau \to \upsilon$, $g : \upsilon \to \phi$ are maps. Then for $a, b \in \tau$,

$$g \circ f(a \wedge b) = g(f(a) \wedge f(b)) = g \circ f(a) \wedge g \circ f(b).$$

$$\tau(a) \leq \upsilon(f(a)) \leq \phi(g(f(a))) \implies \tau(a) \leq \phi(g \circ f(a)).$$

Thus it remains to show that the relation is well-founded.

Lemma 6

Let Q be wqo. Then the set of finite trees labelled by Q, T(Q), is well-founded under homeomorphic embedding.

For a contradiction, suppose not. Then we have a strictly-decreasing chain in $T(Q)$

$$\bar{\tau} := (\tau_1, \tau_2, \tau_3, ...),\ \tau_1 > \tau_2 > \tau_3 > ...$$

Consider the underlying chain of unlabelled trees $t_i := \mathrm{dom}(\tau_i)$. Then since $\mathbb{N}$ is well-founded and $t_i \geq t_j \implies |t_i| \geq |t_j|$, we have a subsequence of trees of equal size. But then, in this subsequence, the maps $t_i \to t_j$ are surjective, and thus order-isomorphisms. Hence we may restrict to the case where $\mathrm{dom}(\tau_i) = \mathrm{dom}(\tau_j) := t$ for all $i, j \in \mathbb{N}$.

Let the vertices of t be $a_1, ..., a_n$, and consider for $i = 1, ..., n$ the sequence

$$\bar{a}_i : \mathbb{N} \to Q : k \mapsto \tau_k(a_i)$$

— that is to say, $\bar{a}_i$ is the sequence of labels at the vertex a_i. Since Q is wqo, by Lemma 1 there is a subsequence $\bar{\tau}_1 \subseteq \bar{\tau}$ such that the corresponding subsequence of $\bar{a}_1$ is non-decreasing. Inductively, if $\bar{\tau}_i \subseteq \bar{\tau}$ is such that the corresponding subsequence of $\bar{a}_j$ is non-decreasing for all $j \leq i$, by Lemma 1 there is a subsequence $\bar{\tau}_{i+1} \subseteq \bar{\tau}_i$ such that the corresponding subsequence of $\bar{a}_{i+1}$ is also non-decreasing.

Then the subsequence $\bar{\tau}_n$ is non-decreasing at every vertex a_i, and so is non-decreasing as a sequence of labelled trees. But it is a subsequence of the decreasing sequence $\bar{\tau}$, which is a contradiction. Hence in fact $T(Q)$ is well-founded under homeomorphic embedding. □

Now that we know $T(Q)$ is a well-founded quasiorder, we can make use of the Minimal Bad Sequence Lemma.

Proving Kruskal's Tree Theorem

For a contradiction, suppose $T(Q)$ is not wqo. Then since $T(Q)$ is a well-founded quasiorder we can take an MBS $\bar{t} : \mathbb{N} \to T(Q)$. As Q is quasiordered, the sequence $\text{root}(\bar{\tau}) : \mathbb{N} \to Q$ has a non-decreasing subsequence $\text{root}(\bar{\tau})_u$ by Proposition 1 (iii). Consider the corresponding sequence $\bar{\tau}_u$ in $T(Q)$, and define for each i the set A_i of branches at the children of the root of $\tau_{u,i}$. Define also

$$A := \bigcup_{i \in \mathbb{N}} A_i;$$

then for all $\rho \in A$, $\rho \in A_i$ for some $i \implies \rho < \tau_{u,i}$. Thus by the MBS Lemma A is wqo.

Moreover, by Proposition 4 (ii) $A^{(<\omega)}$ is also wqo. So we have a good pair $A_i \leq A_j$, which is to say a non-decreasing function

$$f : A_i \to A_j.$$

Since $\rho \leq f(\rho)$ for all $\rho \in A_i$, we have maps $h_\rho : \rho \to f(\rho)$. This lets us define a map $h : \tau_{u,i} \to \tau_{u,j}$ as follows:

- $h(\mathrm{root}(\tau_{u,i})) := \mathrm{root}(\tau_{u,j})$,
- $h|_\rho := h_\rho$ for each branch $\rho \in A_i$.

But this means $\tau_{u,i} \leq \tau_{u,j}$, contradicting the fact that $\bar{\tau}$ is bad. Hence $T(Q)$ is wqo.

Well-foundedness of ε_0

It was shown by Gentzen in 1936 that the Peano axioms are proven consistent by primitive recursive arithmetic along with the statement

$$WO(\varepsilon_0) := \text{the ordinal } \varepsilon_0 \text{ is well-ordered.}$$

In this way we know that (if PA is consistent) PA cannot prove $WO(\varepsilon_0)$. Indeed, since PA interprets primitive recursive arithmetic, such a proof would imply that PA proves its own consistency, which is false by Gödel's second incompleteness theorem.
We will show that Kruskal's tree theorem implies $WO(\varepsilon_0)$, and so is independent of Peano Arithmetic.

Tree representation of ordinals less than ε_0

Every ordinal less than ε_0 may be represented uniquely in its Cantor Normal Form:

$$\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + ... + \omega^{\alpha_n},$$

where $\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_n$ are finitely-many ordinals, each strictly less than α.
Recursively expanding out the α_i in Cantor Normal form until nothing remains but 0 and ω^x yields a very tree-like structure:

$$\omega^{\omega \cdot 2+1} + 3 = \omega^{\omega^{\omega^0}+\omega^{\omega^0}+\omega^0} + \omega^0 + \omega^0 + \omega^0$$

and indeed this is the essence of how we will encode ordinals up to ε_0 as finite trees.

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If T is the set of finite trees, we define $F : \varepsilon_0 \to T$ as follows:

- we define $F(0)$ to be the singleton tree (call it 1_T), and
- given trees $F(\alpha_i)$ for $0 \leq i \leq n$ and $\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + ... + \omega^{\alpha_n}$, $F(\alpha)$ is the tree with branches $F(\alpha_0), ..., F(\alpha_n)$ joined to a single root.

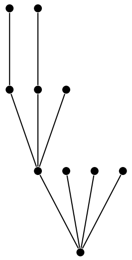

Figure 5: The tree corresponding to $\omega^{\omega \cdot 2+1} + 3$.

Facts about $F : \varepsilon_0 \to T$

- F is a bijection, and
- If $t \leq u$ as trees under homeomorphic embedding, then $F^{-1}(t) \leq F^{-1}(u)$ as ordinals.

The proof of these statements is somewhat involved, but is done by recursively defining its inverse $G : T \to \varepsilon_0$ in terms of the 'height' of a tree (which is the maximum size of an under-set of a vertex).

- $G(1_T) = 0$, and
- if $\mathrm{ht}(t) = k > 0$, let $S := \{s_0, ..., s_n\}$ be the set of the branches at the children of the root. Order them so that

$$G(s_0) \geq G(s_1) \geq ... \geq G(s_n);$$

note that G is already defined on the s_i since they each have height at most $k - 1$. Then set

$$G(t) = \omega^{G(s_0)} + ... + \omega^{G(s_n)}.$$

Kruskal's Tree Theorem proves $WO(\varepsilon_0)$

Take an arbitrary sequence of ordinals below ε_0

$$\bar{\alpha} = (\alpha_0, \alpha_1, \alpha_2, ...).$$

By the bijection $G : T \to \varepsilon_0$, for each of these ordinals there is a unique tree t_i with $G(t_i) = \alpha_i$, giving a corresponding sequence $\bar{t} = (t_0, t_1, t_2, ...)$. But then by Kruskal's tree theorem, there is a good pair $t_i \leq t_j$, which yields a pair $\alpha_i \leq \alpha_j$. Hence $\bar{\alpha}$ is not a strictly-decreasing sequence. So ε_0 is well-founded. □

Corollary 7

Kruskal's theorem is not provable in PA.

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